Robert P. Gilles

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The Cooperative Game Theory of Networks and Hierarchies

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The Cooperative Game Theory of Networks and Hierarchies

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Preface

Textbooks on game theory have mainly been rather short on the theory of games in characteristic function form, also known as “cooperative” game theory. In the standard textbooks of Myerson (1991) and Fudenberg and Tirole (1991), the treatment of cooperative game theory has been limited to a single chapter. Only slightly more attention for cooperative games has been offered in the graduate text of Osborne and Rubinstein (1994). The main texts on cooperative game theory are Ichiishi (1983, 1993), Owen (1995), and Peleg and Sudhölter (2007). With the exception of Ichiishi (1993) these texts emphasize the mathematical foundations of the theory rather than its applications to study certain abstract social and economic phenomena. In this text it is my goal to discuss these mathematical foundations of cooperative game theory as well as some applications of these cooperative game theoretic concepts to construct abstract theories of network formation and the functioning of hierarchical authority situations.

Traditionally cooperative game theory has focussed on two fundamental equilibrium notions in the characteristic function form, the *Core* and the *value*. The Core and related equilibrium concepts are founded on the description of power that coalitions of players are potentially able to exercise in the process of allocating collective benefits to the individual players. Different Core-like concepts reflect the potential differences in the power exercised by the different coalitions. These equilibrium concepts are essentially models of how coalitions might affect bargaining or negotiation processes to allocate collectively generated values. Usually these equilibrium concepts identify collections of allocations that are stable against manipulation by various coalitions of players, where such “manipulation” is defined through appropriate rules in the equilibrium concept. The Core is the simplest of these equilibrium notions and is based on the threat of *any* coalition to abandon the negotiation process completely and to divide the value it can generate by itself among its members. This notion is called “blocking”. A Core allocation is now one that cannot be blocked by any coalition of players.

Value theory, on the other hand, refers to an axiomatic equilibrium concept introduced seminally by Shapley (1953), which caused a revolution in the perception of equilibrium analysis. The value is a uniquely constructed allocation rule that satisfies certain desirable properties or “axioms”. This axiomatic approach first describes the desirable properties of an allocation rule and the subsequent analysis shows the uniqueness of the allocation rule founded on these selected axioms. The original Shapley axioms form a mix of properties describing the power of coalitions to influence the allocation process as well as imposing the (relative) fairness of the allocation of resources to individual players.

Hart and Mas-Colell (1989) seminally introduced a further advance in cooperative game theory, by proposing the reduction of each characteristic function form game to a single number. Surprisingly, this important contribution showed that the proposed *potential function* is closely related to Shapley's axiomatic value. This allows an axiomatic characterization of this potential function as well as the Shapley value. In this text I also discuss recent contributions to potential theory that introduce potentials for alternative value functions and related share functions.

I use the fundamental notions of the Core, the Value and the Potential function in the analysis of social communication networks and the exercise of authority in hierarchical organizations. The applications discussed in this text are mainly based on some recent developments in the literature on the cooperative game theoretic analysis of social networks and hierarchical authority structures. I focus this volume mainly on the contributions made by myself and many of my co-authors. Here I refer in particular to my work in collaboration with René van den Brink, Guillermo Owen, Jean Derks, Gerard van der Laan, and the game theory group at Tilburg University, in particular Stef Tijs, Peter Borm, Henk Norde and Marco Slikker.

The cooperative approach to social networks is developed in an extensive literature emanating from the seminal contribution by Myerson (1977). In this text I limit myself to the discussion of directed communication networks rather than the usually assumed undirected communication networks among players. Directed networks also have numerous applications, including the representation of domination situations such as authority structures and sports competitions.

From directed communication situations it is a small step towards hierarchical authority structures. This is the subject of the last chapter in this text. In this chapter I consider the consequences of the exercise of authority in the form of veto power on the productive values that can be genrated by the participating players. In particular, the chapter considers the Shapley-like values that emerge in these authority situations.

Acknowledgements

This text is founded on joint work with several co-authors. In the subsequent chapters I report my joint research with René van den Brink, Guillermo Owen, Jean Derks, Stef Tijs, Henk Norde, and Marco Slikker. In particular I want to point out the cooperation with René van den Brink as the source of numerous ideas and concepts discussed throughout this text. Without his contributions in numerous fields in (applied) cooperative game theory—axiomatic value theory in particular—this text would not have been written. Several concepts discussed in this text are our joint

creations, in particular the β-value for directed networks and the permission values for hierarchical authority situations.

Also, I thank several of my graduate students at Virginia Tech for their feedback on this manuscript. In particular, I thank Zhengzheng Pan for carefully proofreading most of the text. All remaining errors are my own.

Belfast, UK
September 2009

Robert P. Gilles

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Chapter 1
Cooperative Game Theory

I prefer to think of *Game theory* as a set of tools to describe and analyze social interactive decision situations. This viewpoint leads to several important implications. First, game theory is neither a single theory nor a unified body of knowledge nor a single research program in the sense of Lakatos (1978). Rather it is a collection of a variety of subfields and approaches, each representing a possibly fundamentally different approach to the description of a certain social interactive decision situation.

Second, as the notion of a social interactive decision situation defines the subject matter of game theory, it thereby determines the foundations of each of these subfields in game theory. As such, social interactive decision making as the subject matter of game theory unifies the large variety of tools and theories that make up game theory. There emerges a common understanding of what entails an interactive decision situation. These common features are the following:

- There are *multiple* decision makers involved within a social decision situation.[1]
- Each individual decision maker in principle controls certain decisions within the situation at hand. The potential choices related to these decisions are usually called *actions*. This in turn implies that each individual decision maker has the control over a set of multiple actions to choose from at each decision moment that the decision maker has to execute some choice or action.[2]
- The decisions are *interactive* in the sense that the chosen actions by the different decision makers determine the resulting outcome within the social decision situation. If any of the individual decision makers chances the selected action, in principle the resulting outcome will be affected.

These common features allow us to introduce some well-accepted game-theoretic terminology. A social interactive decision situation is known as a *game*. The decision makers are called the *players* within that game. Each player is assumed to have

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[1] It is easy to see that if we have only one decision maker (player), then there is simply no interaction possible. Instead one then arrives at a standard decision problem.

[2] If a decision maker would only control a set consisting of a single action for a certain decision moment, she would actually not make any decision. She could therefore be omitted as a relevant decision maker at that moment in the decision situation.

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control of at least one decision moment in the course of the game. At each decision moment under her control, a player selects from a well-described set of multiple *actions*.

Game theory has its roots in recreational mathematics.[3] After the first mathematical theorem in game theory in the form of an insight to the nature of the game of chess (Zermelo, 1913) and preliminary work on probabilistic games by Borel (1921), von Neumann (1928) established the first fundamental theorem on the existence and nature of equilibria in zero-sum games in the form of his *minimax theorem*. Under von Neumann's guidance the theory developed into a much broader field with applications in economics and political science. von Neumann (1938) Ultimately Oskar Morgenstern joined von Neumann to lay down the developed theory in their joint seminal text von Neumann and Morgenstern (1953). Over time, game theory became much more serious in its applications, but the playful terminology remained.

This description of the common elements that are studied in game theory allow us to develop the theory in more detail. Namely, in every approach or theory, a game always has to include a description of the players, the structure of the game, the decision moments under the control of each player and the sets of actions that each player at each decision moment has under her control.

In this chapter I first discuss the most prevalent representation of a game in game theory, namely the "normal form". Subsequently I describe a representation known as the "characteristic function form" that can be derived from the normal form description. The characteristic function game form reduces the normal form to its barest essential, namely the collective values that can be generated by any group of players in that game. Throughout this text, the characteristic function form is used as the prime descriptor of games as social interaction situations.

1.1 Aims and Outline of the Book

At the onset of this book I discuss the main game theoretical tools that have been developed to analyze the characteristic function form. The main reason to focus on the characteristic function form is that there have been developed relatively few textbooks, which are fully devoted to this game form. This stands in contrast to the normal form and related game theoretic tools such as the extensive game form. Numerous texts have been devoted to these representations of interactive decision situations. However, I believe that the analysis of interactive decision making using the characteristic function form is very fruitful and has resulted in important insights in socio-economic institutions such as the firm and other authority-based organizations such as government organizations.

[3] For a comprehensive overview of the early history of game theory I refer to Dimand and Dimand (1996) and Schwalbe and Walker (2001).

The second main aim of this text is to give an overview of recent work on using the characteristic function form to understand the functioning and formation of social networks and hierarchical authority organizations. I bring together the insights from a number of recent publications on the analysis of the functioning of social networks throughout the text, but in particular in Chapter 5. I limit myself to the discussion of so-called "directed" social networks, which represent hierarchical relations between players such as authority situations and relational structures with informational asymmetries. That chapter is completely devoted to the analysis of directed communication networks and ranking systems.

Closely related to these contributions on the functioning of social networks are the recent developments in the analysis of the consequences of the exercise of authority. Hierarchies are used throughout our contemporary societies, in particular to direct decision making in hierarchically structured production organizations, usually known as *firms*. From attempts in the economics literature a purely normal form approach to analyzing behavior in a firm is rather unsatisfactory and very difficult. This research has resulted into a field in economics that is also known as the *theory of the firm*. In Chapter 6 I show that the use of characteristic function forms might enlighten the effects of the exercise of hierarchical authority on production values of a firm's divisions. This in turn allows more complex to be analyzed more effectively using normal form reasonings combined with characteristic function form tools.

Next I sketch the subjects discussed in this and the next chapters of this text.

In the remainder of this chapter I introduce the normal as well as the characteristic function game form in mathematical detail. My emphasis will be on the characteristic function form, which is developed in full detail, including several decomposition methods to make its analysis possible.

A *characteristic function* describes the values that groups or coalitions of decision makers or players can generate or claim in some interactive decision situation. In this regard, the characteristic function reduces an interactive decision situation to its barest minimum: the values that can be generated by the players. this excludes the behavioral aspects of the interactive decision situation and puts the characteristic function form of a game firmly within the realm of cooperative game theory as the theory addressing the selection of a binding agreement among the players. In this regard the aim of the analysis of the characteristic function form is to identify and formulate a binding agreement that is to the satisfaction of all players and all feasible player coalitions.

In the cooperative game theoretic literature there have emerged two main approaches to addressing this problem. First, the analysis turned to identifying those contracts to which no coalition of players can make a legitimate objection. Here, one understands the legitimacy of an objection to be founded on the ability of such a coalition to propose an alternative agreement which is preferable for all members of the coalition. This lies at the foundation of the notion of the *Core* of a characteristic function form game.

In Chapter 2 of this text I discuss the Core and related concepts. The main classical problem related to the Core is under which conditions there exist such agreements that are stable to any objection, i.e.; the Core is not empty. Exact conditions have been formulated, resulting into the Bondareva–Shapley Theorem. In Chapter 2 I also discuss applications of these ideas to situations in which not all coalitions can legitimately form. This is mainly the case when authority is exercised to prevent certain subordinates from joining certain coalitions or if there are communication deficiencies that prevent certain groups of players to form a functioning coalition. For many of these situations the literature has provided full descriptions of the resulting Cores. I discuss several of these characterizations.

A completely different approach to finding "acceptable" binding agreements is forwarded through *value theory*. Here the emphasis is on the properties that a certain proposed agreement possesses. A number of these properties can be formulated and an agreement can be identified that satisfies these properties. In Chapter 3 I discuss the most important values and their properties. The seminal contribution to this field is that of the *Shapley value* (Shapley, 1953), which combines properties describing coalitional power and fairness. The basic idea is that the net gains that a coalition can generate are divided equally among its members. Since its inception, there have been developed numerous axiomatic characterizations of the Shapley value and it has emerged as an exceptionally important concept that balances coalitional power and fairness in a very intricate fashion.

The fundamental properties of the Shapley value describing coalitional power and fairness can be applied to situations in which certain coalitions cannot form or remain powerless. This generalized concept is known as the *Myerson value* and applies exactly to those situations resulting from the exercise of hierarchical authority or the effects of deficiencies in social networks. Chapter 5 on social communication networks and Chapter 6 on hierarchical authority organizations develop specific formulations of the (general) Myerson value for the analysis of these structures on decision making.

The concept of a "value" can also be viewed from a completely different perspective. Indeed, a value essentially summarizes the information contained in a characteristic function into a vector of numbers, one for each of the players in the game. In this fashion, a higher dimensional structure is thus condensed into a vector. In Chapter 4 I discuss the extension of this reductionist approach and introduce the notion of a *cooperative potential*. A potential is a function that reduces the information of a characteristic function form into a single number. That is, the information of the class of all characteristic function forms is reduced to a single potential function. From this function one can reconstruct characteristic functions for arbitrary games and, amazingly, the Shapley values for all of these games. In fact, the cooperative potential is closely related to the Shapley value and, therefore, to the more general Myerson value. The theory of the cooperative potential can be viewed as a central achievement in the theory of the characteristic function form. The seminal contribution to the theory of the cooperative potential was Hart and Mas-Colell (1989).

Finally, Chapters 5 and 6 use the analytical framework provided by the theory of the Core, Value theory, and the cooperative potential to understand the effects of relational structures on interactive decision situations. In Chapter 5

I consider directed networks imposed on a population of players. First, I discuss methods to measure the domination of certain players in such a structured population over other players. This results into the analysis of (axiomatic) *power measures*. Next, I consider the extension of standard value theory to situations with a structured player population. This results into the development of so-called hierarchical values.

In Chapter 6 I consider the consequences of the exercise of authority in a hierarchically structured population of players. In these hierarchical structures, superiors are assumed to have veto power over the actions pursued by their subordinates. The ultimate sanction is to exclude a subordinate completely from the productive activities in the hierarchical organization. I discuss two models of the exercise of such authority. For both models, I give characterizations of the related Myerson values, also known as *permission values*. The main insights from this analysis can then be applied to understand the functioning of hierarchical production organizations such as firms.

1.2 Game Forms

There are two fundamentally different approaches to the description of an interactive decision situation. The first approach is based on the absolute absence of any binding agreements between the decision makers in these interactive decision situations. This is also known as *non-cooperative* game theory.[4]

The non-cooperative approach fits very well with many applications of a social interactive decision situation as described by a game. Indeed, it assumes that each player in a game is driven by a well-formulated goal. This goal is formalized as the player's *payoff function*. This function assigns to each outcome resulting from a selection of individual actions a payoff or "utility". Each player now optimizes her payoff by selecting actions that are under her control. How this is determined is actually the subject matter of non-cooperative game theory. As such, non-cooperative game theory is the most pristine expression of the principle of *methodological individualism* that lies at the foundation of most of contemporary microeconomics.

The second fundamental approach is known as *cooperative* game theory and allows players to write binding contracts. This changes the analysis and interpretation of a game radically. Indeed, if binding agreements can be written, all players collectively will pursue the maximization of the total wealth that can be generated within the social decision situation at hand. A binding contract then determines how this generated wealth is distributed among the various players in this interactive decision situation.

[4] I touch upon this approach only in a limited fashion in this chapter; this text does not intend to include a comprehensive overview of non-cooperative game theory. For much more elaborate treatments of non-cooperative game theory I refer to the standard game theory texts such as Myerson (1991), Osborne and Rubinstein (1994), Vega-Redondo (2003), and Osborne (2004).

Thus, the main objective of cooperative game theory is to determine a "just" or "well-supported" contract between all players to divide the total wealth generated collectively. Such a contract can be based on pure bargaining power or solely on fairness considerations or mixtures of both power and fairness.

Although non-cooperative and cooperative game theory are fundamentally separated through the acceptance of the hypothesis of allowing binding contracts, in technical terms the various representations of interactive decision situations span these two approaches. Here I present a short overview of the three main representation forms[5] that are employed in game theory:

The extensive form The extensive form is the most complete description of an interactive decision situation that is developed in game theory.[6] It fully describes the sequential decision making processes and the resulting outcomes and their evaluation. In this regard the extensive form contains very detailed information and consequentially is very specific to the decision situation at hand.
The detailed sequential structure of the extensive form has also some significant drawbacks. Indeed, any analysis of such a specific description can only be specific as well. This is shown in the work done on equilibrium concepts for the extensive form. There is a multitude of equilibrium concepts, each geared towards a specific set of properties that make each equilibrium concept applicable for a certain type of decision situation described by extensive form games.

The normal form A reduction of the detailed information in the extensive form is at the foundation of the normal form. In such games only the strategic interaction structure of the decision situation is fully represented. The mathematical structure of the normal is much simpler than the extensive form; on the other hand, the normal form lacks the sequential information that is at the foundation of the extensive form.
The limitation of the normal form to the strategic structure of the decision situation has a significant advantage that the analysis is much more transparent. In particular, the number of equilibrium concepts is limited since only a few of such concepts are meaningful. More powerful insights can therefore be established in the context of the normal form.

The characteristic function form As discussed above, the third main game form discards even more information and only considers wealth levels that can be assigned to the various groups or *coalitions* of players in an interactive decision situation. This is in fact the preferred game form to describe cooperative

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[5] For a complete description of these main game forms I also refer to Myerson (1991) and Vega-Redondo (2003). It would be out of the range of these lecture notes to dwell in length on the mathematical representation of the extensive game form, in particular.

[6] In this context I mention the social situation form developed by Greenberg (1990), which is some respects is more general than the extensive form and in other respects is less general. This representation form has not been accepted widely in game theory, though.

games in which binding agreements are investigated. Indeed, given that there are binding agreements, it is no longer of importance how these wealth levels are achieved, but only how these wealth levels are allocated to the players in the interactive decision situation. Hence, the selection of actions becomes irrelevant in favor of a description of the contract among the players in the game.

The major strength of the characteristic function form is its simplicity; this is also its major weakness. The interactive decision situation itself is no longer represented by this game form, but rather it represents the negotiation process to allocate the generated wealth that follows the decision process itself.

The simplicity of the characteristic function form allows us to model the negotiation process through an appropriately designed equilibrium concept. Hence, the game itself acts as an "input" to the equilibrium concept, which becomes the main tool. This is in contrast to the extensive form—and to an extend the normal form as well—in which the description of the interactive decision situation is the main analytical tool. In cooperative game theory, the analysis and comparison of the various equilibrium concepts becomes the focus of the theoretical development.

In the remainder of this section I discuss the normal as well as the characteristic function form representations of an interactive decision situation. Here I focus strictly on the representative nature of these game forms. In the next section I shall subsequently turn to the use of the characteristic function form as an independent tool of analysis of coalitional bargaining processes. For this I no longer require that an underlying interactive decision situation is identified. Instead, characteristic function form games are employed to study abstract bargaining processes.

1.2.1 The Normal Form

In the normal form representation of an interactive decision situation each player only has exactly one decision moment. At that single decision moment all players simultaneously select an action from a given set of feasible actions available for that player. Formally, if each player selects a single action, the list of all actions chosen is also denoted as an *action tuple*.[7]

An outcome results from the chosen actions, expressing the interactive nature of the decision situation at hand. This outcome is evaluated by each player individually through a "utility" or "payoff" function. The payoff function represents the

[7] The notion of a tuple is equivalent to the mathematical concept of a vector in a linear vector space. However, since the sets of feasible actions to the players might be non-Euclidean, game theorists prefer to use the vocabulary introduced here.

well-defined objective that each player is assumed to have.[8] Such utility functions are known as "payoff" functions in game theory.

Definition 1.1 A triple $\langle N, A, \pi \rangle$ is a non-cooperative game in *normal form* if

(i) $N = \{1, \dots, n\}$ is a set of players,
(ii) $A = A_1 \times A_2 \times \cdots \times A_n$ is a set of action tuples, where for every player $i \in N$, the set A_i represents the actions under her control, and
(iii) $\pi : A \to \mathbb{R}^N$ is a vector function that assigns for every player $i \in N$ and every action tuple $a \in A$ a payoff $\pi_i(a) \in \mathbb{R}$.

To illustrate the notion of the normal form we develop a very simple application. It concerns a trade situation among three persons.

Example 1.2 Consider a trade situation with three traders. Trader 1 has a book for sale. His *reservation value*[9] is \$10. Traders 2 and 3 are potential buyers of the book. Their reservation values are \$150 and \$160, respectively. In the ensuing simultaneous bargaining procedure, trader 1 puts in an ask and traders 2 and 3 simultaneously make a bid. The book is traded to the highest bidder among traders 2 and 3, provided that trader 1 puts in an ask lower or equal than the highest bid. The book is traded at the average of the ask and the highest bid. If both buyers bid the same, they trade with equal probability with the seller.

We can formalize this very simple bargaining situation as follows. The player set is $N = \{1, 2, 3\}$. The action players' sets are given by $A_1 = A_2 = A_3 = \mathbb{R}_+$. Finally, regarding the resulting payoffs, consider $a = (a_1, a_2, a_3) \in A = \mathbb{R}^3_+$, then the payoff function for the seller (Trader 1) is constructed to be

$$\pi_1(a) = \begin{cases} \frac{1}{2}(a_1 + \max\{a_2, a_3\}) - 10 & \text{if } a_1 \leqq \max\{a_2, a_3\} \\ 0 & \text{if } a_1 > \max\{a_2, a_3\} \end{cases}$$

while the payoff functions of the two buyers (Traders 2 and 3) are respectively given by

$$\pi_2(a) = \begin{cases} 150 - \frac{1}{2}(a_1 + a_2) & \text{if } a_1 \leqq a_2 \text{ and } a_2 > a_3 \\ 75 - \frac{1}{4}(a_1 + a_2) & \text{if } a_1 \leqq a_2 \text{ and } a_2 = a_3 \\ 0 & \text{otherwise} \end{cases}$$

[8] From this formulation is should be clear that the players are assumed to be fully *rational*. Indeed, each player has a well-defined objective in the form of the utility function given. Each player is now assumed to pursue her objective to the fullest extend within the context of the interactive decision situation described in the normal form game.

[9] The reservation value is a threshold value for the participation of that player in the game. For a buyer, the reservation value is the highest price at which he is willing to buy the item, usually equal to the monetary equivalent of the utility of the item under consideration. For a seller, the reservation value is the lowest price at which she is willing to part with the book in question, usually the procurement or production cost of the item under consideration.

and

$$\pi_3(a) = \begin{cases} 160 - \frac{1}{2}(a_1 + a_3) & \text{if } a_1 \leqq a_3 \text{ and } a_3 > a_2 \\ 80 - \frac{1}{4}(a_1 + a_3) & \text{if } a_1 \leqq a_3 \text{ and } a_3 = a_2 \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

The formulation of these payoff functions is a consequence of the reservation values as well as the bargaining procedure described above. ■

Within the normal form, each player is assumed to pursue her individual goal by rationally optimizing her payoff. As such this is an expression of *methodological individualism*. In turn, this results in the notion of a stable state in the game, also known as an equilibrium state. I do not discuss equilibrium concepts in the normal form here, but instead refer to standard texts on game theory—among which Myerson (1991), Osborne and Rubinstein (1994) and Vega-Redondo (2003)—for a comprehensive and extensive discussion of equilibrium concepts.

Instead I discuss the method in which a normal form representation can be converted into a characteristic function form representation seminally introduced by von Neumann and Morgenstern (1953).

1.2.2 The Characteristic Function Form

Next consider the case that cooperation between players is permitted and that binding agreements can be written, implemented and enforced. This changes the analysis rather drastically.

First, the players—as the main decision making entities in the game—are now considered to negotiate with each other to determine a binding agreement between them. To fully develop the different possibilities within a game for cooperation among players we have to address what groups of players can achieve collectively. The basic entities in the analysis, therefore, are these feasible groups of players. These groups are indicated as *coalitions*.

Second, the attainability of binding agreements makes it possible for these groups of players to create leverage in the bargaining process. Indeed, if a certain group of players assesses that it does not receive what it perceives to be able to get by itself, then this group might decide to abandon the negotiations with the other players in the game and pursue an alternative allocation by itself.

The result is that the negotiation process converges to some steady state or equilibrium in which none of the groups of players have any incentives to object against the agreement that is proposed. It is this form of stability that should be the subject of any meaningful theory of the cooperative approach. The resulting allocations form the so-called "Core" of the game. Variations on this notion are subject to an extensive literature.

This methodology has been at the foundation of game theory since the origination of many concepts in von Neumann and Morgenstern (1953). In fact, John von Neumann strongly believed that the ultimate goal of game theory should be to describe theories and models that would represent the processes that in real life occur between humans in social situations. These real-life processes usually contain non-cooperative elements as well as the formulation of binding agreements. Until now this ultimate goal has been quite illusive to game theory.

We introduce the terminology that helps us to set up the theory of cooperation between players.

Definition 1.3 Let N be some finite set of players. Formally a group of players $S \subset N$ is called a *coalition*. Specifically, $\varnothing$ is denoted as the empty coalition and the player set N itself is denoted as the grand coalition.

The collection of all coalitions is denoted by $2^N = \{S \mid S \subset N\}$.

The use of the terminology introduced in this definition requires some clarification. It is proper to think of a "coalition" in a different sense than a mere "group of players". Indeed, a coalition is endowed with an instilled sense of purpose and is essentially assumed to be able to formulate and execute collective actions. This in turn implies that a coalition is a group of players endowed with a collective decision mechanism or a *governance* structure. Hence, institutions and behavioral norms are in place as well as a communication structure that allow these players to plan, formulate and execute collective actions.

As described above what is attainable for an arbitrary coalition is central to any concept of equilibrium within the context of cooperative game theory. There are two different approaches to assign values to a coalition. These correspond to an *optimistic* and a *pessimistic* outlook on what a coalition can achieve—or more properly *attain*—in the setting of a non-cooperative game in normal form.

For this purpose consider a normal form non-cooperative game $\langle N, A, \pi \rangle$ and a coalition $S \in 2^N$ with $S \neq \varnothing$. For the coalition S we define $A_S = \prod_{i \in S} A_i$ as the action set at the disposal of the players in coalition S and $F_S \colon \mathbb{R}^S \to \mathbb{R}$ as its utility or payoff *aggregator*. Then we can formally develop the two approaches to assigning a value to coalition S:

The pessimistic view: The Maximin Aggregation Method. In this approach a coalition S considers the value $v^\alpha(S)$ that it can produce as the highest aggregated utility level that it can generate in response to the coordinated actions that the other players in the game select. This leads to the "maximin" aggregation method:

$$v^\alpha(S) = \max_{x_S \in A_S} \min_{y_{N \setminus S} \in A_{N \setminus S}} F_S \left(\pi_i(x_S, y_{n \setminus S})_{i \in S} \right) \tag{1.2}$$

This definition implies that coalition S chooses a best response to the worst action that the complement coalition $N \setminus S$ can implement against it. Thus, a coalition is assigned what it can obtain against optimally behaving non-members.

The optimistic view: The Minimax Aggregation Method. In this approach a coalition S considers the value $v^\beta(S)$ that it can create for itself as the highest aggregated utility level generated in response to the actions that the other players in the game select. Hence, S as a collective always selects a coordinated best response to the non-members actions. This is laid down in the "minimax" aggregation method:

$$v^{\beta}(S) = \min_{y_{N\setminus S}\in A_{N\setminus S}} \max_{x_S\in A_S} F_S\left(\pi_i(x_S, y_{n\setminus S})_{i\in S}\right) \tag{1.3}$$

This definition implies that the complement coalition $N \setminus S$ selects a worst response to any coordinated action undertaken by S. Hence, a coalition is assigned what it can generate maximally in a hostile world.

Originally these two methods of converting non-cooperative normal form games into characteristic functions were developed in the seminal text by von Neumann and Morgenstern (1953). The exact methodologies developed above are slight variations of the original methods. Other aggregation methods are also discussed in Myerson (1991, Section 9.2).

My approach is based on the notion of an aggregator—or "social welfare function" applied in social choice theory—that generates a single value from payoffs for each member of the coalition in question. This implies the application of a hypothesis that the payoffs for the various players can be compared and that through the aggregator they can be "added" and that the total payoff can then be re-distributed.

The generated functions above are also known as *characteristic functions of the normal form game* $\langle N, A, \pi\rangle$. Formally, $v^{\alpha}\colon 2^N \to \mathbb{R}$ is known as the *α-characteristic function* of non-cooperative game $\langle N, A, \pi\rangle$ and $v^{\beta}\colon 2^N \to \mathbb{R}$ is known as the *β-characteristic function* of non-cooperative game $\langle N, A, \pi\rangle$. A pair (N, v) with $v\colon 2^N \to \mathbb{R}$ is also known as a cooperative game with *transferable utilities* or a game in characteristic function form.

In general the optimistic viewpoint generates higher coalitional values than the pessimistic viewpoint. Hence, in general, $v^{\alpha}(S) \leqq v^{\beta}(S)$ for all coalitions S and all regular aggregators F. This is illustrated with the following example.

Example 1.4 Let $N = \{1, 2\}$ and consider the following 2×2-matrix game for these two players:

1╲2	**L**	**R**
U	1,0	0,1
D	0,1	1,0

In this matrix the first reported number in each field in the matrix is the payoff to player 1, π_1, while the second reported number is the payoff to player 2, π_2.

Consider the so-called *utilitarian* aggregator defined by $F^u_S(\pi_S) = \sum_{i\in S}\pi_i$ for every coalition $S \subset N$. Using the definitions above we compute now:

$$v^{\alpha}(1) = v^{\alpha}(2) = 0 \text{ and } v^{\alpha}(12) = 1$$
$$v^{\beta}(1) = v^{\beta}(2) = 1 \text{ and } v^{\beta}(12) = 1$$

It is clear that the optimistic view gives both players a rosy outlook on the game and the belief that they both can achieve a high payoff. On the other hand, the pessimistic world view lets both players anticipate a lower payoff level. In some respects both viewpoints are incorrect; one player always gets a high payoff and one player always gets a low payoff. ■

We discuss some examples to discuss the role of the aggregator before turning to the description of a description of the general introduction of games in characteristic function form without relying on a specified aggregator.

Example 1.5 Consider the non-cooperative trade game developed in Example 1.2. Within the context of this normal form game we consider two different aggregators.

First, we consider the *utilitarian* aggregator defined by $F^u_S(\pi_S) = \sum_{i \in S} \pi_i$ for every coalition $S \subset N$. For this particular game we have that both the optimistic and the pessimistic viewpoint lead to the same payoff values: $v_u(1) = v_u(2) = v_u(3) = 0$, $v_u(12) = 140$, $v_u(13) = 150$, $v_u(23) = 0$ and $v_u(123) = 150$.

Second, we consider the *Rawlsian* aggregator defined by $F^r_S(\pi_S) = \min_{i \in S} \pi_i$ for every coalition $S \subset N$. For this particular game we have that the optimistic and the pessimistic viewpoint lead to different payoff values. I give here the values for the optimistic formulation: $v^\beta_r(1) = v^\beta_r(2) = v^\beta_r(3) = 0$, $v^\beta_r(12) = 70$, $v^\beta_r(13) = 75$, $v^\beta_r(23) = 0$ and $v^\beta_r(123) = 46\frac{2}{3}$. For the Rawlsian aggregator the players pursue a situation that is as egalitarian as possible for all members of the coalition. In particular, if $S = N = 123$, then one of the two buyers is always out of luck and fully determines the total payoff generated by the grand coalition. This is contrary to the total payoff generated by the utilitarian aggregator. ■

These simple examples show that there is a variety of ways to generate and evaluate outcomes for coalitions in cooperative situations. In the remainder of these lecture notes we limit ourselves to the essentials of these descriptors.

For this generalized setting I will develop various properties and discuss some basic properties, in particular on the normalization of the representation of the coalitional values.

1.3 Cooperative Games

In the discussion thus far I only considered characteristic functions that were derived from non-cooperative normal form game through a variety of aggregation methods. Hence, I limited myself to the application of the characteristic function form as a direct description of an interactive decision situation. However, von Neumann and Morgenstern (1953) already considered characteristic function games as an independent category of games. These “abstract” or “general” games in characteristic function form simply describe an interactive decision situation from the perspective of the generated collective payoffs only. Each coalition of players is assigned a payoff or “worth” and equilibrium concepts are developed to describe how these values are allocated to the members of these coalitions.

Using the seminal definition given by von Neumann and Morgenstern (1953), I develop the notion of a (general) cooperative game in full detail.

Definition 1.6 The pair (N, v) is a *cooperative game* (in characteristic function form) if N is a finite player set and $v\colon 2^N \to \mathbb{R}$ is a characteristic function that assigns to every coalition $S \subset N$ an attainable payoff $v(S)$ such that $v(\varnothing) = 0$.

For every player set N we denote by $\mathcal{G}^N$ the class of all characteristic functions on N.

A cooperative game in characteristic function form represents the attainable payoffs for the different coalitions directly as collective payoff values. There is no description of the actual, feasible actions available to the individual players in the interactive decision situation. Rather, only the attainable payoff values are described.

In principle such payoff values as generated by cooperating players are *transferable* in the sense that payments can be made to the individual members of a coalition. Such representations are therefore also known as "side payment games" or "games with transferable utilities," for short TU-games. The definition of an allocation of these transferable values is rather straightforward in this setting; it simply is a vector of payments $x \in \mathbb{R}^N$ to individuals such that $\sum_{i \in N} x_i = V(N)$, where $v(N)$ represents the total generated wealth by the collective of all players in the game.

1.3.1 Basic Properties of Cooperative Games

Here, I will consider some basic properties or requirements that one can impose on cooperative games in characteristic function form. These properties are usually required to establish certain results about the existence and nature of stable binding agreements and value concepts.

The first standard property is that of superadditivity,[10] which requires that unions of coalitions generate higher values than the separated coalitions, i.e.; there is assumed a synergy between cooperating players.

Formally, a cooperative game (N, v) is denoted as *superadditive* if for all $S, T \in 2^N$ with $S \cap T = \varnothing$:

$$v(S) + v(T) \leqq v(S \cup T). \tag{1.4}$$

A special form of superadditivity is *additivity*, requiring that for all $S, T \in 2^N$ with $S \cap T = \varnothing$ it holds that

$$v(S) + v(T) = v(S \cup T). \tag{1.5}$$

[10] In economics "superadditivity" refers to the presence of some form of *increasing returns to scale*. In this particular case it refers that the marginal contribution of a single player to the value generated by some coalition is larger than what this player can generate by herself.

A further reduction of the superadditivity notion is also seminally considered von Neumann and Morgenstern (1953). A cooperative game (N, v) is called *constant-sum* if for every coalition $S \subset N$:

$$v(S) + v(N \setminus S) = v(N). \tag{1.6}$$

Another fundamental property is monotonicity. Formally, a cooperative game (N, v) is *monotone* if for all $S, T \in 2^N$:

$$S \subset T \text{ implies } v(S) \leqq v(T). \tag{1.7}$$

A cooperative game (N, v) is *symmetric* if the value $v(S)$ only depends on the number of players $|S|$ in the coalition S. Hence, there is some function $f : \mathbb{N} \to \mathbb{R}$ such that $v(S) = f(|S|)$ for all $S \subset N$.

A cooperative game (N, v) is *simple* if for each coalition $S \subset N$ we have either $v(S) = 0$ or $v(S) = 1$.

Finally, consider two cooperative games $v, w \in G^N$. Then the *sum* of v and w is defined by the cooperative game $v + w \in G^N$ given as

$$(v + w)(S) = v(S) + w(S) \text{ for every coalition } S \subset N. \tag{1.8}$$

All properties discussed thus far are invariant under summation in the sense that the sum of two superadditive games is superadditive as well, the sum of two additive games is additive as well, and the sum of two monotone games is monotone as well.

1.3.2 The Standard Basis

Next, I investigate how cooperative games can be decomposed into certain simple games. These decompositions characterize a cooperative game as a configuration of games with straightforward payoff structures. For technical analysis, the latter is an important tool to arrive at powerful conclusions regarding cooperative games and certain equilibrium concepts.

Before discussing such decompositions of cooperative games, I show that the space of cooperative games is in fact a multi-dimensional Euclidean vector space. The class of all cooperative games $\mathcal{G}^N$ is a linear vector space in the sense that for all games $v, w \in \mathcal{G}^N$ and scalars $\lambda, \mu \in R$ we can define the linear combination $\lambda v + \mu w \in \mathcal{G}^N$ by

$$(\lambda v + \mu w)(S) = \lambda v(S) + \mu w(S) \tag{1.9}$$

Under these algebraic operations, it is indeed the case that $\mathcal{G}^N$ is a linear vector space with the *null game* $\eta \in \mathcal{G}^N$—defined by $\eta(S) = 0$ for all coalitions $S \subset N$—as a proper zero element in $\mathcal{G}^N$. Since all values that are attained in cooperative games are real numbers, the linear vector space $\mathcal{G}^N$ is even a Euclidean space.

Regarding the dimensionality of $\mathcal{G}^N$ for $N = \{1, \ldots, n\}$, the space $\mathcal{G}^N$ is equivalent to a Euclidean space of dimension $2^n - 1$. Note that a characteristic function is a listing of numbers of length 2^n. However the requirement that $v(\varnothing) = 0$ puts a single constraint on this listing, leading to the dimensionality as stated. Hence, $\mathcal{G}^N \backsim \mathbb{R}^{2^n-1}$.

The standard basis of a Euclidean space is the one consisting of unit vectors. In the space of cooperative games $\mathcal{G}^N$ this corresponds to the set of so-called standard basis games. For every coalition $S \subset N$ with $S \neq \varnothing$ we define the *standard basis game* $b_S \in \mathcal{G}^N$ given by

$$b_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{if } T \neq S \end{cases} \tag{1.10}$$

Now $\mathcal{B} = \{b_S \mid S \subset N,\ S \neq \varnothing\}$ is a set of standard basis elements in the linear vector space $\mathcal{G}^N$ consisting of $2^n - 1$ characteristic functions of the corresponding standard basis games. Indeed we can now write any $v \in \mathcal{G}^N$ as

$$v = \sum_{S \neq \varnothing} v(S)\, b_S. \tag{1.11}$$

Although $\mathcal{B}$ is a basis of $\mathcal{G}^N$, it turns out not be to be very useful. We will revisit this issue again a bit later.

1.3.3 The Unanimity Basis

The game space $\mathcal{G}^N$ has an alternative basis, the so-called unanimity basis denoted by $\mathcal{U}$, which is in fact much more useful than the standard basis $\mathcal{B}$. The unanimity basis fulfills a central role in the discussion of the most important equilibrium concepts in cooperative game theory and was seminally introduced to cooperative game theory by Shapley (1953).

For every coalition $S \subset N$ we define the *unanimity game* $u_S \in \mathcal{G}^N$ by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subset T \\ 0 & \text{otherwise} \end{cases} \tag{1.12}$$

The following proposition is at the foundation of the property that the set of unanimity games in fact forms a basis of $\mathcal{G}^N$.

Proposition 1.7 *Every game $v \in \mathcal{G}^N$ can be written as*

$$v = \sum_{S \neq \varnothing} \Delta_v(S)\, u_S, \tag{1.13}$$

where $\Delta_v(S) \in \mathbb{R}$ *is the* ***Harsanyi dividend*** *of coalition S in game v defined by*

$$\Delta_v(S) = \sum_{T \subset S} (-1)^{s-t}\, v(T) \tag{1.14}$$

in which $s = |S|$ *and* $t = |T|$.

Proof This proof follows the outline developed in Owen (1995, p. 263). Let $T \subset N$ be any coalition. Then we have that

$$\begin{aligned}\sum_{S \subset N} \Delta_v(S) u_S(T) &= \sum_{S \subset T} \Delta_v(S) = \\ &= \sum_{S \subset T} \left(\sum_{R \subset S} (-1)^{s-r} v(R) \right) = \\ &= \sum_{R \subset T} \left(\sum_{S \subset T:\, R \subset S} (-1)^{s-r} \right) v(R).\end{aligned}$$

Consider the inner parenthesis in the last expression. For every subcoalition $R \subset T$ and every value of $r \leqq s \leqq t$, there will be $\binom{t-r}{t-s}$ sets S with s elements such that $R \subset S \subset T$. Hence, the inner parenthesis may be replaced by

$$\sum_{s=r}^{t} \binom{t-r}{t-s} (-1)^{s-r}.$$

But this exactly the binomial development of $(1-1)^{t-r}$. Hence,this expression is zero for all $r < t$ and unity for $r = t$. Thus,

$$\sum_{S \subset N} \Delta_v(S)\, u_S(T) = v(T)$$

for all $T \subset N$ as required. ■

The main conclusion from Proposition 1.7 is that the collection of unanimity games forms a proper basis of the game space $\mathcal{G}^N$.

Corollary 1.8 *The collection of unanimity basis games* $\mathcal{U} = \{u_S \mid S \subset N,\ S \neq \varnothing\}$ *forms a basis of* $\mathcal{G}^N$. *The collection* $\mathcal{U}$ *is also known as the* ***unanimity basis*** *of* $\mathcal{G}^N$.

The Harsanyi dividends Δ_v of a cooperative game $v \in \mathcal{G}^N$ play an important role in the development of the theory of solution concepts for these games. (This concept was seminally introduced by Harsanyi (1963) in his discussion of the Shapley value.) It is easier to compute these dividends recursively with the following procedure.

(i) $\Delta_v(\varnothing) = 0$ and

(ii) for every coalition $S \neq \varnothing$:

$$\Delta_v(S) = v(S) - \sum_{R \subsetneq S} \Delta_v(R). \tag{1.15}$$

We conclude our discussion with a simple example.

Example 1.9 Consider the utilitarian cooperative trade game developed in Example 1.5. We list the various values of this game in the following table:

S	$v(S) = v_u(S)$	$\Delta_v(S)$
$\varnothing$	0	0
1	0	0
2	0	0
3	0	0
12	140	140
13	150	150
23	0	0
123	150	–140

The table above implies that we can write this utilitarian trade game as

$$v_u = v = 140\, u_{12} + 150\, u_{13} - 140\, u_{123} \tag{1.16}$$

The decomposition of the game v_u into unanimity games will turn out to be rather useful in discussing axiomatic value theory in Chapter 3. ■

The definition of addition of games leads to a standard equivalence concept on the space of all cooperative games $\mathcal{G}^N$.

Definition 1.10 Two games $v, w \in \mathcal{G}^N$ are S-*equivalent*—or simply "equivalent"—if there exist a positive number $\lambda > 0$ and n real constants $\mu_1, \ldots, \mu_n \in \mathbb{R}$ such that

$$v = \lambda\, w + \sum_{i \in N} \mu_i\, u_i \tag{1.17}$$

where $u_i \in \mathcal{G}^N$ is defined by $u_i(S) = 1$ if $i \in S$ and $u_i(S) = 0$ if $i \notin S$.

From the definition it follows that two games $v, w \in \mathcal{G}^N$ are equivalent if for all $S \subset N$:

$$v(S) = \lambda\, w(S) + \sum_{i \in S} \mu_i \tag{1.18}$$

If two games are equivalent, one can obtain one from the other by simply performing a linear transformation on the utility spaces of the various players.

1.3.4 Essential Games and Imputations

In this section I consider the transformation of a cooperative game in an equivalent form that is easier to analyze. We use S-equivalency to establish such relationships. Throughout our discussion we consider games that have sufficient structure to perform non-trivial transformation. These games are called “essential”.

Definition 1.11 A game $v \in \mathcal{G}^N$ is *essential* if $v(N) > \sum_{i \in N} v(\{i\})$.

In an essential game there is a positive difference between the minimal values that each player can attain individually—described by the value $v(\{i\})$—and the total value that can be attained by the whole population of players—given by $v(N)$. This positive difference is the wealth that can be allocated among the players in the game.

Essential games can be “normalized” in the sense that they can be scaled to lower payoff levels without affecting the solutions of that particular game. Within the Euclidean space $\mathcal{G}^N$ this corresponds to a simple linear transformation. The main normalization that we employ is the so-called $(0, 1)$-normalization of a game.

Definition 1.12 A game $v \in \mathcal{G}^N$ is $(0, 1)$-*normal* if $v(\{i\}) = 0$ for every individual player $i \in N$ and $v(N) = 1$.

The following proposition states a relatively straightforward property. A proof is therefore omitted.

Proposition 1.13 *If $v \in \mathcal{G}^N$ is essential, then v is S-equivalent to exactly one* $(0, 1)$-*normal game $w_v \in \mathcal{G}^N$ given by*

$$w_v(S) = \frac{1}{\varepsilon(v)} \left[v(S) - \sum_{i \in S} v(\{i\}) \right] \tag{1.19}$$

for every coalition $S \subset N$, where

$$\varepsilon(v) = v(N) - \sum_{i \in N} v(\{i\}) > 0. \tag{1.20}$$

The main objective of cooperative game theory is to develop models and solution concepts that address the “proper” division of the excess value $\varepsilon(v) > 0$ generated by the players in an essential game $v \in \mathcal{G}^N$. The subsequent discussion in the following chapters debates this issue from various perspectives.

The main tool in the discussion of allocating the generated surplus $\varepsilon(v) > 0$ over the players is that of an *imputation*. The allocation of the total generated value $v(N)$ among the n players in the cooperative game v can be represented by an n-dimensional Euclidean vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.[11] Here x_i denotes the

[11] Throughout these lecture notes I follow the standard convention on the use of inequalities. Let $x, y \in \mathbb{R}^n$. Now $x \geqq y$ iff $x_i \geqq y_i$ for all $i = 1, \ldots, n$; $x > y$ iff $x \geqq y$ and $x \neq y$; finally, $x \gg y$ iff $x_i > y_i$ for all $i = 1, \ldots, n$. Here the expression “iff” stands for “if and only if”.

payment to player i in allocation x. An imputation is an allocation that satisfies two conditions:

Definition 1.14 An *imputation* in the cooperative game $v \in \mathcal{G}^N$ is a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^N$ satisfying

$$\sum_{i \in N} x_i = v(N) \qquad \text{(Efficiency)}$$

$$x_i \geqq v(\{i\}) \text{ for every player } i \in N \qquad \text{(Individual rationality)}$$

The set of all imputations in the game v is denoted by $I(v) \subset \mathbb{R}^N$.

An imputation is an allocation that satisfies two fundamental properties:

Efficiency The efficiency requirement imposes that the total collective wealth generated in a cooperative game v is divided among its constituents. Since $v(N)$ represents this total collectively wealth, the condition in Definition 1.14 indeed formulates this requirement. The term "efficiency" refers to the fact that there cannot be divided more than the collective N is able to generate.

Individual rationality Each individual player $i \in N$ views the value $v(\{i\})$ as fully attainable by herself without the cooperative of any other player in the collective decision situation. In this regard the individual rationality condition imposes that each individual player has property rights to this individually attainable value without any limitations. This is formalized in the individual rationality condition.

This discussion allows us to reformulate the notion of an imputation. Indeed, an imputation is an allocation of the total collective value over the constituting players in the cooperative game such that each player receives her individually attainable wealth level. It is obvious that the set of imputations $I(v)$ is non-trivial if and only if the game v is essential.

For the special case of three players, $N = \{1, 2, 3\}$, the imputation set can be fully characterized. It is easier to restrict the discussion to $(0, 1)$-normal games. Under that condition the imputation set is simply the two-dimensional *unit simplex*, defined by

$$I(v) = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3_+ \;\middle|\; x_1 + x_2 + x_3 = 1 \right\}. \tag{1.21}$$

Figure 1.1 on page 20 depicts this set in a two-dimensional representation. The unit simplex is a triangular shape and each imputation now is a point in the interior or on the boundary of this triangle. The three sides of the unit simplex—also denoted as "facets" of the simplex—are fully determined by the three equations

$$\text{Facet 1: } x_1 + x_2 = 1$$
$$\text{Facet 2: } x_1 + x_3 = 1$$
$$\text{Facet 3: } x_2 + x_3 = 1$$

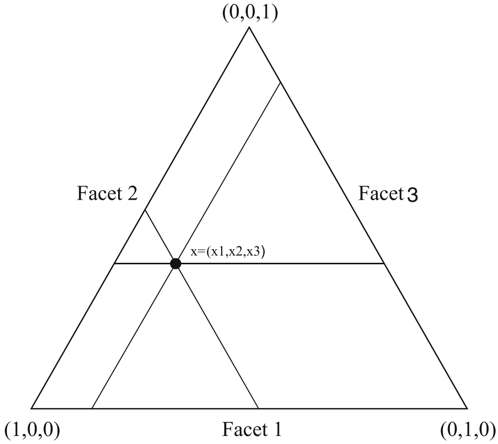

Fig. 1.1 Imputation set $I(v)$ of a $(0,1)$-normalized three player game

The facets of the unit simplex describe imputations in which exactly one of the three players does not receive any side payment. In Facet 1, player 3 is excluded in the sense that his payoff is always zero. Similarly for the other two facets.

1.4 Multilinear Extensions

Owen (1972) introduced an alternative approach to the description of a cooperative game. He developed the notion of a "multilinear" extension of a game. This is a mathematical device that makes computations of certain equilibrium concepts of cooperative games more transparent; in particular, it allows the algorithmic computation of solution concepts. The notion of the multilinear extension was further developed in Owen (1975) and Owen (1995). In this section I define the concept of multilinear extension and provide an introductory discussion. In Chapter 3 I apply the multilinear extension to compute the Shapley value of a cooperative game and in Chapter 6 I apply the concept to compute (hierarchical permission) values of hierarchically structured cooperative games.

The idea of the multilinear extension is to extend the characteristic function of a cooperative game to an enhanced class of objects, which can be indicated as *probabilistic coalitions*. These probabilistic coalitions are essentially probabilistic variables that represent the probability a certain coalition might form, given the probabilities of each player that he will participate in that coalition. In particular, if player $i \in N$ participates in some coalition with probability $0 \leqq p_i \leqq 1$, then coalition $S \subset N$ forms exactly with probability

$$\prod_{i \in S} p_i \cdot \prod_{j \notin S} (1 - p_j)$$

noting that $1 - p_i$ is the probability that player i does *not* participate in a coalition that might form. The multilinear extension of a characteristic function now assigns to every vector of probabilities $(p_i)_{i \in N}$ the expected value that is generated by the probabilistic coalition under those probabilities. This interpretation of the multilinear extension is formalized in the discussion following Example 1.19.

Consider a cooperative game $v \in \mathcal{G}^N$. A game can also be interpreted as a function that assigns to every coalition $S \in 2^N$ the value $v(S)$. Hence, $v\colon 2^N \to \mathbb{R}$. The multilinear extension of a game v now extends the domain 2^N of the characteristic function v to the N-dimensional unit cube $[0,1]^N$. Here I note that the collection of all coalitions 2^N in N is analytically equivalent to the binary space $\{0,1\}^N$, where for every coalition $S \subset N$ the vector $\chi_S \in \{0,1\}^N$ given by

$$\chi_S(i) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

is a full characterization of S; it exactly describes whether a player i is a member of S or not.

The multilinear extension performs the extension from the binary membership space $\{0,1\}^N$ to the unit cube $[0,1]^N$ in a particular fashion. Namely, it preserves the basic linear structure of the cooperative game. The multilinear extension is a function of n variables that is linear in every variable.

Definition 1.15 Let $v \in \mathcal{G}^N$. The *multilinear extension (MLE)* of the cooperative game v is the real-valued function $E_v\colon [0,1]^N \to \mathbb{R}$ defined by

$$E_v(x) = \sum_{S \subset N} \left[\prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \right] v(S) \tag{1.22}$$

for every $x = (x_1, \ldots, x_n) \in [0,1]^N$.

Before we turn to the question whether the MLE of a cooperative game is well-defined, we go through some simple examples.

Example 1.16 Let $N = \{1,2,3\}$. Consider $v = u_{12}$, the unanimity game of coalition $\{1,2\}$. Then from the definition it follows immediately that

$$\begin{aligned} E_v(x) &= \sum_{S \subset N} \left[\prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \right] v(S) = \\ &= \sum_{S \subset N\colon\ 12 \subset S} \left[\prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \right] = \\ &= x_1 x_2 (1 - x_3) + x_1 x_2 x_3 = x_1 x_2. \end{aligned}$$

Similarly, consider $w = u_1$. Then we determine that

$$E_w(x) = \sum_{S \subset N:\ 1 \in S} \left[\prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \right] =$$
$$= x_1(1 - x_2)(1 - x_3) + x_1 x_2 (1 - x_3) + x_1 x_3 (1 - x_2) + x_1 x_2 x_3 =$$
$$= x_1(1 - x_3) - x_1 x_2 (1 - x_3) + x_1 x_2 (1 - x_3) + x_1 x_3 = x_1$$

From these computations it is clear that there is a significant regularity to the MLEs of unanimity games, which will be explored next. ■

We can formulate the findings of the previous example as a general property. A proof of the next assertion is relegated to the problem section of this chapter.

Lemma 1.17 *Let $S \subset N$. If $u_S \in \mathcal{G}^N$ is the unanimity game of the coalition S, then its multilinear extension is given by*

$$E_{u_S}(x) = \prod_{i \in S} x_i \tag{1.23}$$

Next we turn to the question whether the MLE of a cooperative TU-game is indeed well-defined.

Proposition 1.18 *Let $v, w \in \mathcal{G}^N$.*

(a) The multilinear extension $E_v \colon [0,1]^N \to \mathbb{R}$ of the game v is indeed an extension of v in functional sense and it is unique as such.
(b) For all scalars $\alpha, \beta \in \mathbb{R}$, the game $\alpha v + \beta w \in \mathcal{G}^N$ has the multilinear extension given by $\alpha E_v + \beta E_w$.

Proof For the proof of assertion (a) I follow the outline given in Owen (1995, p. 268–269).

First, that the MLE of a cooperative game is an extension of the function v is clear from the following discussion. Indeed, let $S \subset N$ and define $\chi(S) \in [0,1]^N$ by $\chi_i(S) = 1$ if and only if $i \in S$. Then from the definition it is clear that

$$E_v(\chi_S) = v(S).$$

This indeed implies that the function E_v is an extension of the function v in the standard sense from real function theory.

To show uniqueness, we re-write the MLE as a polynomial of n variables as follows

$$E_v(x) = \sum_{S \subset N} C_S \prod_{j \in S} x_j \tag{1.24}$$

where C are 2^n constants. Then the Equation (1.24) reduces to

$$E_v(\chi_S) = \sum_{T \subset S} C_T,$$

which in turn through $E_v(\chi_S) = v(S)$ implies that

$$\sum_{T \subset S} C_T = v(S) \text{ for all } S \subset N.$$

This describes a system of 2^n linear equations with 2^n unknowns C_S, $S \subset N$. We know that $\{\Delta_v(S) \mid S \subset N\}$ is a solution to this system of equations.[12] Furthermore, all equations are independent.[13] Thus, this system of linear equations has a unique solution. This in turn implies that the MLE of the game v is uniquely defined.

The proof of assertion (b) is immediate from the definition of the MLE of a cooperative game and, therefore, omitted. ■

From Lemma 1.17 and Proposition 1.18(b) we can now re-write the MLE of a cooperative game $v \in \mathcal{G}^N$ as follows

$$E_v(x) = \sum_{S \subset N} \Delta_v(S) \cdot \left\{ \prod_{i \in S} x_i \right\} \tag{1.25}$$

This is a handier expression than the formal definition of E_v, which we used earlier in our discussion.

Example 1.19 Consider the cooperative game developed in the series of examples throughout this chapter. The given payoff values in the table in Example 1.9 now implies together with the formulation above that

$$E_v(x) = 140x_1x_2 + 150x_1x_3 - 140x_1x_2x_3.$$

From this MLE we can in turn compute various equilibria and solution concepts, which we will address in the next two chapters. ■

We now discuss a *probabilistic interpretation* of the MLE of a cooperative game. Let $v \in \mathcal{G}^N$ and consider its MLE E_v. Let $\beth$ be a random coalition in N. We define $\mathcal{E}_i$ as describing the event that $i \in N$ is a member of $\beth$. Now let $x_i = \text{Prob}\,(\mathcal{E}_i)$.

Assuming that $\{\mathcal{E}_i \mid i \in N\}$ are independent events, we can write for any coalition $S \subset N$ that

[12] I remark that this implies the assertion stated in Lemma 1.17.

[13] Formally, the system of linear equations can be described through a matrix of coefficients. This matrix is nonsingular. Hence, the system of linear equations has a unique solution.

$$\text{Prob}\,(\beth = S) = \prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i)$$

and, therefore, the expected payoff of random coalition $\beth$ is given by

$$\mathbb{E}\,[v(\beth)] = \sum_{S \subset N} \text{Prob}(\beth = S) \cdot v(S) = E_v(x).$$

We may conclude from this discussion that the MLE of a cooperative game can be interpreted as describing the expected value of a randomized coalition. In other words, if each player $i \in N$ has a probability x_i of joining a random coalition, then this random coalition can command $E_v(x)$ as its expected amount of utility.[14]

This probabilistic interpretation of the MLE of a cooperative game allows us to formulate another fact:

Proposition 1.20 *Let $v \in G^N$ be a constant-sum game and let E_v be its multilinear extension. Then for any x:*

$$E_v(e - x) = v(N) - E_v(x), \tag{1.26}$$

where $e = (1, 1, \ldots, 1) \in \mathbb{R}^N_+$ is the vector of ones.

Proof Using the probabilistic interpretation of the MLE of a cooperative game, it can be verified that

$$E_v(e - x) = \mathbb{E}[v(N \setminus \beth)]$$

where $\beth$ is the random coalition from our previous discussion. Indeed, $1 - x_i$ is the probability that player i is a member of the complement of the coalition $\beth$, being $N \setminus \beth$.

Now it immediately follows that

$$\begin{aligned} E_v(e - x) + E_v(x) &= \mathbb{E}[v(N \setminus \beth)] + \mathbb{E}[v(\beth)] = \\ &= \mathbb{E}[v(N \setminus \beth) + v(\beth)] = \\ &= v(N), \end{aligned}$$

since for an arbitrary coalition $S \subset N$ it holds that $v(N \setminus S) + V(S) = v(N)$. This implies the assertion. ■

[14] This probabilistic interpretation of the MLE of some cooperative game allows us to construct an alternative proof of the assertion stated in Lemma 1.17. This is the subject of a problem in the problem section of this chapter.

1.5 Problems

Problem 1.1 Consider Example 1.5. Now impose that all three traders have a certain wealth constraint $W_i > 200$, $i = 1, 2, 3$, above which they cannot bid or ask, i.e.; the players' strategy sets are modified such that for each player $a_i \in A_i \equiv [0, W_i]$.

(a) Show that the imposition of wealth levels for the three traders has no consequences for the derived game in characteristic function for for the utilitarian aggregator.
(b) Carefully check the computations in this example of the different coalitional payoffs for the Rawlsian aggregator. In particular, check the characteristic function for both the pessimistic (α) and optimistic (β) cases. Does the imposed wealth constraints change the derived characteristic functions in any way?
(c) Next consider the aggregator defined by $F_S(\pi_S) = \max_{i \in S} \pi_i$ for every coalition $S \subset N$. Compute the resulting coalitional values for this alternative aggregator for both the pessimistic (α) and optimistic (β) cases. In particular, show that the imposed wealth constraints are essential for the feasibility of this aggregator.

Problem 1.2 Consider the Rawlsian cooperative game theoretic representation of the trade situation considered in Example 1.5. Give a unanimity decomposition of this Rawlsian game using a methodology similar to the one used in Example 1.9 for the utilitarian cooperative representation.

Problem 1.3 Myerson (1991) discusses a simple behavioral alternative to the two aggregation methods founded on considerations debated in von Neumann and Morgenstern (1953). Consider a normal form game $\langle N, A, \pi \rangle$ and a coalition $S \subset N$ and its complement $N \setminus S$. As usual we define $A_S = \prod_{i \in S} A_i$ as the set of action tuples for coalition $S \subset N$. Now, consider action tuples $\bar{x}_S \in A_S$ and $\bar{x}_{N \setminus S} \in A_{N \setminus S}$ such that

$$\bar{x}_S \in \arg\max_{x_S \in A_S} \sum_{i \in S} \pi_i \left(x_S, \bar{x}_{N \setminus S} \right) \tag{1.27}$$

$$\bar{x}_{N \setminus S} \in \arg\max_{x_{N \setminus S} \in A_{N \setminus S}} \sum_{j \in N \setminus S} \pi_j \left(\bar{x}_S, x_{N \setminus S} \right) \tag{1.28}$$

Myerson's characteristic function is now defined as v^m with

$$v^m(S) = \sum_{i \in S} \pi_i \left(\bar{x}_S, \bar{x}_{N \setminus S} \right) \quad \text{and} \quad v^m(N \setminus S) = \sum_{j \in N \setminus S} \pi_j \left(\bar{x}_S, \bar{x}_{N \setminus S} \right).$$

(a) Determine under which conditions on the normal form game, in particular the action sets A_i, $i \in N$, and the payoff functions π_i, $i \in N$, this aggregation method is well-defined and proper.

(b) Explain why this method is denoted as the *defensive-equilibrium* aggregation method by Myerson in his discussion.

Problem 1.4 Harsanyi (1963) introduced an alternative method to aggregate payoffs from a normal form game into a characteristic function form. This *Harsanyi aggregation method* is founded on a Nash's bargaining model (Nash, 1950). Myerson (1991) refers to this method as the "rational-threats representation".

Consider a normal form game $\langle N, A, \pi \rangle$ such that the action set A_i is compact and convex in some Euclidean space for every player $i \in N$. As usual we define $A_S = \prod_{i \in S} A_i$ as the set of action tuples for coalition $S \subset N$. Now, for every coalition $S \subset N$ and its complement $N \setminus S$ let there be action tuples $\bar{x}_S \in A_S$ and $\bar{x}_{N\setminus S} \in A_{N\setminus S}$ such that

$$\bar{x}_S \in \arg\max_{x_S \in A_S} \left[\sum_{i \in S} \pi_i \left(x_S, \bar{x}_{N\setminus S} \right) - \sum_{j \in N\setminus S} \pi_j \left(x_S, \bar{x}_{N\setminus S} \right) \right] \tag{1.29}$$

$$\bar{x}_{N\setminus S} \in \arg\max_{x_{N\setminus S} \in A_{N\setminus S}} \left[\sum_{j \in N\setminus S} \pi_j \left(\bar{x}_S, x_{N\setminus S} \right) - \sum_{i \in S} \pi_i \left(\bar{x}_S, x_{N\setminus S} \right) \right] \tag{1.30}$$

Harsanyi's characteristic function is now introduced as v^h, where

$$v^h(S) = \sum_{i \in S} \pi_i \left(\bar{x}_S, \bar{x}_{N\setminus S} \right) \quad \text{and} \quad v^h(N \setminus S) = \sum_{j \in N\setminus S} \pi_j \left(\bar{x}_S, \bar{x}_{N\setminus S} \right).$$

Show that this aggregation method is properly defined and that for each normal form game satisfying the stated conditions the Harsanyi characteristic function is well-defined.

Problem 1.5 Consider the imputation set $I(v)$ of a $(0,1)$-normalized three player game v as depicted in Fig. 1.1.

(a) It is shown in Fig. 1.1 that each interior imputation has six non-orthogonal projections on the three facets of the simplex. These projections are parallel along the three facets of the unit simplex. Give a complete characterization of these six projections.
(b) There are also three orthogonal projections of the imputation x on the three facets of the unit simplex. Give a graphical representation of these three projections and characterize them analytically as well.

Problem 1.6 Provide a detailed and exact proof of the assertion stated in Lemma 1.17. In this proof you should use a probabilistic argument based on the interpretation of the MLE developed prior to Proposition 1.20.

Problem 1.7 Consider $N = \{1, \ldots, n\}$ and a vector of non-negative values $w = (w_1, \ldots, w_n)$. Define the cooperative game v by

$$v(S) = \left(\sum_{i \in S} w_i \right)^2 .$$

(a) Give the decomposition of this game into unanimity games using the appropriate Harsanyi dividends.
(b) Determine the multilinear extension (MLE) of the given game. (*Hint:* Use the probabilistic approach to the MLE.)
(c) Determine the (0, 1)-normalization of the given game.

Chapter 2
The Core of a Cooperative Game

The main fundamental question in cooperative game theory is the question how to allocate the total generated wealth by the collective of all players—the player set N itself—over the different players in the game. In other words, what binding contract between the players in N has to be written? Various criteria have been developed.

In this chapter I investigate contracts that are stable against deviations by selfish coalitions of players. Here the power of the various coalitions in the negotiation process is central. Hence, the central consideration is that coalitions will abandon the negotiation process if they are not satisfied in their legitimate demands. This is at the foundation of the concept of the *Core* of a cooperative game and subject of the present chapter.

The Core stands in contrast with (Shapley) value theory, which is explored in the next chapter. Value theory aims at balancing criteria describing negotiation power with fairness considerations. This approach leads to the fair allocation of assigned values to the different coalitions.

In certain respects these two approaches to the fundamental question of cooperative game theory conflict with each other. Core allocations are founded on pure power play in the negotiation process to write a binding agreement. It is assumed that coalitions will not give up on their demands under any circumstance. This is very different in comparison with the Value. In Value theory it is assumed that coalitions will forego demands based on the exercise of pure power, but instead abide by a given set of behavioral principles including a certain reflection of power and fairness. This leads to situations in which the Value is not a Core allocation, while in other situations the Value is a central allocation in the Core. The first case refers to a conflict between fairness and power considerations, while the second case refers essentially to the alignment of the fairness and power aspects in these two approaches.

On the other hand there is also a very interesting overlap or agreement between the two approaches based on power (the Core) and balancing power and fairness (the Value). The common feature here is that power is a common feature of these two solution concepts. Therefore, it is crucial to understand the presence and exercise of power in allocation processes in a cooperative game. I will explore these aspects in this and the next chapter as well.

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After defining Core allocations and discussing the Core's basic properties, I introduce the existence problem and the well-known Bondareva–Shapley Theorem. The main condition for existence identified is that of *balancedness* of the underlying cooperative game.

An important extension of the Core concept captures the fact that not all groups of individuals actually can form as functional coalitions. This results into the consideration of constraints on coalition formation and collections of so-called "institutional" coalitions which have the required governance structure to make binding agreements. The equivalent existence theorem for these partitioning games is stated as the Kaneko–Wooders theorem. The main condition for existence of the Core under constraints on coalition formation in the context of arbitrary cooperative games is that of *strong balancedness* of the underlying collection of these institutional coalitions.

Subsequently I look at the specific structure of the Core if the institutional coalitions form a lattice. This results in a rather interesting set of applications, which includes the exercise of authority in hierarchical organization structures. Finally, I discuss a number of so-called "Core covers"—collections of allocations that include all Core allocations—including the Weber set and the Selectope.

2.1 Basic Properties of the Core

The fundamental idea of the Core is that an agreement among the players in N can only be binding if every coalition $S \subset N$ receives collectively at least the value that it can generate or claim within the characteristic function form game, which is actually the generated value given by $v(S)$. This leads to the following definition.

Definition 2.1 A vector $x \in \mathbb{R}^N$ is a *Core allocation* of the cooperative game $v \in \mathcal{G}^N$ if x satisfies the efficiency requirement

$$\sum_{i \in N} x_i = v(N) \tag{2.1}$$

and for every coalition $S \subset N$

$$\sum_{i \in S} x_i \geqq v(S). \tag{2.2}$$

Notation: $x \in C(v)$.

We recall the definition of the set of imputations $I(v)$ of the game $v \in \mathcal{G}^N$ given in Definition 1.14. As introduced there, an imputation is an individually rational and efficient allocation in the cooperative game v. This is subsumed in the definition of a Core allocation. Hence, $C(v) \subset I(v)$. Throughout we will therefore also use "Core imputation" for these allocations in $C(v)$.

The notion of the Core of a cooperative game has a long history. The basic idea was already formulated by Edgeworth (1881) in his discussion of trading processes between economic subjects in a non-market trading environment. These trading processes were based on the formulation of a collective trade contract. An allocation of economic commodities has to be stable against re-barter processes within coalitions of such economic actors. Allocations that satisfied this fundamental re-trade immunity are indicated as *Edgeworthian equilibrium* allocations. The set of such Edgeworthian equilibrium allocations is also indicated as the "contract curve" by Edgeworth himself or the "Core of an economy" since the full development of this theory in the 1960s. A full account of this theory is the subject of Gilles (1996).

Upon the development of game theory, and cooperative game theory in particular, the fundamental idea of blocking was formulated by Gillies (1953) in his Ph.D. dissertation. This idea was developed further in Gillies (1959) and linked with the work of Edgeworth (1881) by Shubik (1959). Since Shubik's unification, the theory of the Core—both in cooperative game theory as well as economic general equilibrium theory—took great flight.[1]

The main appeal and strength of the Core concept is that the notion of blocking is very intuitive. It is a proper formalization of the basic power that the various groups of players have in bargaining processes. A coalition is a group of players that has the institutional structure to plan and execute actions, including the allocation of generated value over the members of that coalition. (I discussed this also in Chapter 1.) In this respect, the value $v(S)$ that coalition S can generate is fully attainable. The blocking process now allows coalitions to fully access this attainable value and allocate it to its constituting members.

2.1.1 Representing the Core of a Three Player Game

I develop the representation of the Core of a three player game in detail. Assume that $N = \{1,2,3\}$ and that $v \in \mathcal{G}^N$ is a $(0,1)$-normal game. Note that under these assumptions, there are only three non-trivial coalitions with potential blocking power to be considered, $12 = \{1,2\}$, $13 = \{1,3\}$, and $23 = \{2,3\}$. Each of these three coalitions might execute its power to object against a proposed imputation. Indeed, other coalitions have no potentiality to block: the singleton coalitions only attain zero, while the grand coalition evidently is excluded from potentially blocking a proposed imputation.

We can summarize these observations by applying the definition of the Core to obtain the following set of inequalities that fully describe the Core of the game v:

[1] For more details on the history of the Core equilibrium concept I also refer to Weintraub (1985), Hildenbrand and Kirman (1988), and Gilles (1996).

$$x_1, x_2, x_3 \geqq 0 \tag{2.3}$$

$$x_1 + x_2 + x_3 = 1 \tag{2.4}$$

$$x_1 + x_2 \geqq v(12) \tag{2.5}$$

$$x_1 + x_3 \geqq v(13) \tag{2.6}$$

$$x_2 + x_3 \geqq v(23) \tag{2.7}$$

Recalling that

$$I(v) = \mathcal{S}^2 \equiv \{x \in \mathbb{R}^3_+ \mid x_1 + x_2 + x_3 = 1\},$$

it is easy to see that the Core of the game v is given by

$$C(v) = \{x \in I(v) \mid \text{ Inequalities (2.5)–(2.7) hold }\}.$$

We can therefore conclude that generically the Core of a three player game v in $(0,1)$-normalization is fully determined by the three inequalities (2.5)–(2.7) in relation to the two-dimensional simplex. This is depicted in Fig. 2.1. The three inequalities (2.5)–(2.7) are also known as the *coalitional incentive constraints* imposed by the Core.

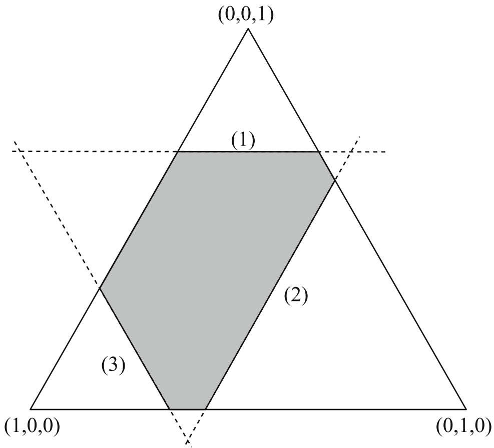

Fig. 2.1 Constructing the Core of a three player $(0,1)$-normal game

In Fig. 2.1 inequality (2.5) is depicted as (1), inequality (2.6) as (2), and inequality (2.7) as (3). The area between these three lines in the two-dimensional simplex is the set of allocations satisfying the coalitional incentive constraints for this game. (Outside the unit simplex of imputations, this area is enclosed by dashed lines.) The Core $C(v)$ is now the set of imputations satisfying these three coalitional incentive constraints; thus, $C(v)$ is the intersection of the depicted triangular area and the unit simplex $I(v)$ of imputations.

In Fig. 2.1 a situation is depicted in which $C(v) \neq \varnothing$. This is the case when there is an inverse triangle created between the three lines (1), (2) and (3). The intersection of this inverse triangle and the two-dimensional unit simplex is now the Core of the $(0, 1)$-normal game under consideration.

Next consider the case depicted in Fig. 2.2 on page 33. In this situation the three inequalities (2.5)–(2.7) do not form an inverse triangle, but rather a regular triangle. This indicates that the three inequalities are exclusive and that there are no imputations that satisfy them. In other words, this situation depicts the case of an empty Core, $C(v) = \varnothing$.

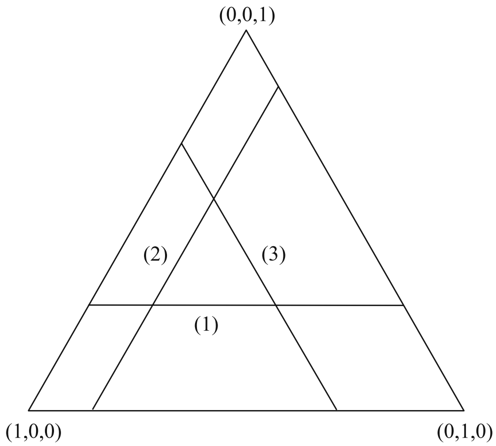

Fig. 2.2 A game with an empty Core

An example of a $(0, 1)$-normal game with three players that satisfies the conditions depicted in Fig. 2.2 is one in which

$$v(12) = v(13) = v(23) = \tfrac{3}{4}.$$

It is easy to verify that, indeed, $C(v) = \varnothing$.

Next we can elaborate on the various situations and shapes of the Core that might result in a three player game when it is not empty. First, we consider the case of a non-empty Core fully within the interior of the space of imputations. Formally, the Core of a game v is called "interior" if

$$\varnothing \neq C(v) \subset \operatorname{Int} I(v), \text{ where } \operatorname{Int} I(v) = \{x \in I(v) \mid x \gg 0\}.$$

An example of such an interior Core is given in Fig. 2.3 on page 34. The Core is well bounded away from the three facets of the two-dimensional unit simplex.

Example 2.2 Consider a $(0, 1)$-normal game with player set $N = \{1, 2, 3\}$ given by $v_1(12) = \frac{1}{2}$ and $v_1(13) = v_1(23) = \frac{3}{4}$. Now the Core is a single, interior imputation:

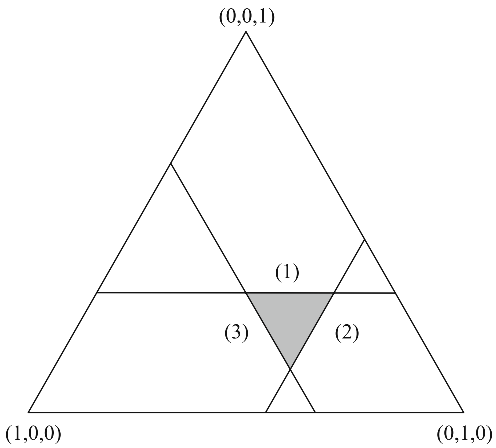

Fig. 2.3 An interior Core

$C(v_1) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$. A simple modification modification of the game v_1 given by $v_2(12) = \frac{3}{8}$ and $v_2(13) = v_2(23) = \frac{3}{4}$ results into an interior Core given by

$$C(v_2) = \text{Conv}\left\{\left(\frac{1}{8}, \frac{1}{4}, \frac{5}{8}\right), \left(\frac{1}{4}, \frac{1}{8}, \frac{5}{8}\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)\right\}$$

The three corner points in this Core are determined by the intersection of two of the three coalitional incentive constraints (1), (2), and (3). ∎

Second, we can consider cases in which the Core is not in the interior of the imputation space, but rather has imputations common with the boundary of the imputation space. The boundary of the imputation space $I(v)$ is defined by

$$\partial I(v) = \{x \in I(v) \mid x_i = 0 \text{ for some } i \in \{1, \dots, n\}\}.$$

The Core of a game v is called "anchored" if

$$C(v) \cap \partial I(v) \neq \varnothing.$$

Examples of an anchored Core are given in the original depiction of the Core of a three player game in Fig. 2.1 as well as the one given in Fig. 2.4 on page 35.

The extreme case of an anchored Core is the one in which the set of imputations itself is equal to the Core. The simplest case with this property is subject of the next example.

Example 2.3 Consider the simple game $v \in \mathcal{G}^N$ with $N = \{1, 2, 3\}$ given by $v(S) = 0$ for all $S \subsetneq N$ and $v(N) = 1$. It is immediately clear that the game v is indeed simple and that the Core is equal to the imputation set, i.e., $C(v) = I(v)$. ∎

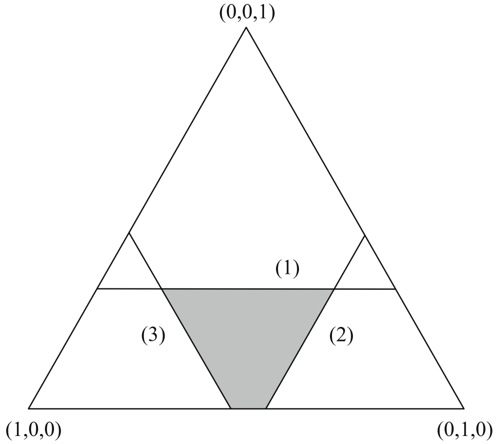

Fig. 2.4 An anchored Core

2.1.2 *The Core and Domination*

There is an alternative approach to the definition of the Core of a cooperative game. This approach is based on the domination of one imputation by another. The notion of domination has its roots in the seminal work on bargaining processes by Edgeworth (1881). Edgeworth considered coalitions of traders exchanging quantities of commodities in an economic bargaining process that is not based on explicit price formation, also known as Edgeworthian barter processes. Later this was interpreted as the "Core of an economy" (Hildenbrand and Kirman, 1988).

The concept of domination is well-known[2] in game theory in general, in particular within the context of normal form non-cooperative games. For cooperative games as well as non-cooperative games, the notion of dominance is essentially equivalent; the payoffs under the various situations are compared and one situation dominates the other if these payoffs are simply higher.

Definition 2.4 An imputation $x \in I(v)$ *dominates* another imputation $y \in I(v)$ with $x \neq y$ if there exists a coalition $S \subset N$ with $S \neq \varnothing$ such that the following two conditions hold

(D.1) $\sum_{i \in S} x_i \leqq v(S)$
(D.2) $x_i > y_i$ for every $i \in S$

An imputation $x \in I(v)$ is *undominated* if there is no alternative imputation $y \in I(v)$ such that y dominates x through some coalition S.

[2] In many texts, including Owen (1995), the definition of the notion of the Core of a cooperative game is simply based on the domination of imputations. The definition employed here—based on the satisfaction of all coalitions in Core allocations—is then treated as a derivative notion. Equivalence theorems are used to express under which conditions these two different approaches lead to the same sets of Core allocations.

The notion of domination is very closely related to the definition of the Core. Indeed, if $x \notin C(v)$, then there exists at least one coalition $S \subset N$ and a partial allocation $y_S \in \mathbb{R}^S_+$ for that particular coalition such that

$$\sum_{i \in S} y_i = v(S)$$
$$y_i > x_i \text{ and } y_i \geqq v(\{i\}) \text{ for all } i \in S \tag{2.8}$$

In the literature this alternative definition of the Core is described by the notion of "blocking". Indeed, the pair (S, y_S) *blocks* the imputation x. Similarly, we say that a coalition $S \subset N$ can block the imputation x if there exists a partial allocation y_S for S such that the conditions (2.8) hold. The Core of the game v is now exactly the set of imputations that cannot be blocked by any coalition $S \subset N$.

Blocking and domination seem rather closely related. However, there is a subtle difference in the sense that domination is defined to be a relational property on the imputation set $I(v)$, while blocking is limited to distinguish Core imputations from non-Core imputations. This subtle difference is sufficient to arrive at different sets of imputations:

Theorem 2.5 *Let $v \in \mathcal{G}^N$ be a cooperative game. Then*

(a) *Every Core allocation $x \in C(v)$ is an undominated imputation.*
(b) *If v is superadditive, then $x \in C(v)$ if and only if x is undominated.*

For a proof of this theorem I refer to the Appendix of this chapter.

To show that the reverse of Theorem 2.5(a) is not valid unless the game is superadditive, I construct the next example.

Example 2.6 Consider the three-player game v given in the following table:

S	$\varnothing$	1	2	3	12	13	23	123
$v(S)$	0	2	2	2	5	5	5	6

Then the imputation set is a singleton given by $I(v) = \{(2,2,2)\} = \{x\}$. By definition this single imputation is undominated. (Indeed, there are no other imputations to dominate it.) However, this single imputation is not in the Core of the game v. Indeed for the coalition $S = \{1,2\}$ it holds that

$$x_1 + x_2 = 4 < 5 = v(S).$$

This implies that coalition S will object to the imputation x even though it is the only feasible allocation of the total generated wealth $v(N) = 6$. ■

2.1.3 Existence of Core Imputations

One of the main questions in the theory of the Core is the problem of the existence of Core imputations. I start the discussion with a simple investigation of games with an empty Core. The following proposition states a sufficient condition under which a cooperative game has an empty Core.

Proposition 2.7 *If $v \in \mathcal{G}^N$ is an essential constant-sum game, then $C(v) = \varnothing$.*

Proof Suppose to the contrary that the Core of v is not empty. Now select $x \in C(v)$. Then for every player $i \in N$

$$\sum_{j \in N \setminus \{i\}} x_j \geqq v(N \setminus \{i\}) \text{ and } x_i \geqq v(\{i\}).$$

Since v is a constant-sum game, it is the case that

$$v(N \setminus \{i\}) + v(\{i\}) = v(N).$$

Hence, from the above,

$$v(N) = \sum_{j \in N} x_j \geqq x_i + v(N \setminus \{i\})$$

implying that $x_i \leqq v(\{i\})$. This in turn implies that $x_i = v(\{i\})$. Therefore, by essentiality,

$$v(N) = \sum_{i \in N} x_i = \sum_{i \in N} v(\{i\}) < v(N).$$

This is a contradiction, proving the assertion. ■

The main problem regarding the existence of Core imputations was seminally solved by Bondareva (1963). However, her contribution was written in Russian and appeared in a rather obscure source. It therefore remained hidden for the game theorists in the West. Independently Shapley (1967) found the same fundamental existence theorem. Since information reached the West that Bondareva (1963) solved the problem first, this fundamental existence theorem has been known as the Bondareva–Shapley Theorem. If information had reached researchers in the West earlier, it might be known now simply as the Bondareva Theorem.

The main concept to understanding the exact requirements for a non-empty Core is that of a *balanced* collection of coalitions.

Definition 2.8 Let $\mathcal{B} \subset 2^N \setminus \{\varnothing\}$ be a collection of non-empty coalitions in the player set N. The collection $\mathcal{B}$ is *balanced* if there exist numbers $\lambda_S > 0$ for $S \in \mathcal{B}$ such that for every player $i \in N$:

$$\sum_{S \in \mathcal{B}:\, i \in S} \lambda_S = 1 \tag{2.9}$$

The numbers $\{\lambda_S \mid S \in \mathcal{B}\}$ in (2.9) are known as the *balancing coefficients* of the collection $\mathcal{B}$.

A balanced collection of coalitions B is *minimal* if it does not contain a proper subcollection that is balanced.

One can interpret balanced collections of coalitions as generalizations of partitions, using the notion of membership weights. Indeed, let $\mathcal{B} = \{S_1, \dots, S_k\}$ be some partitioning of N, i.e., $\cup_{m=1}^{k} S_m = N$ and $S_m \cap S_l = \varnothing$ for all $m, l \in \{1, \dots, k\}$. Then the collection $\mathcal{B}$ is balanced for the balancing coefficients $\lambda_S = 1$ for all $S \in \mathcal{B}$.

The next example gives some other cases of balanced collections, which are not partitions of the player set.

Example 2.9 Consider $N = \{1, 2, 3\}$. Then the collection $\mathcal{B} = \{12, 13, 23\}$ is balanced with balancing coefficients given by $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Indeed, every player is member of exactly two coalitions in $\mathcal{B}$, implying that the sum of the balancing coefficients for each player is exactly equal to unity.

We remark that for the three player case the collection $\mathcal{B}$ is in fact the *unique* balanced collection of coalitions that is not a partition of the set N.

Next let $N' = \{1, 2, 3, 4\}$ and consider $\mathcal{B}' = \{12, 13, 14, 234\}$. Then $\mathcal{B}'$ is balanced with balancing coefficients given by $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$. To verify this, we note that player 1 is member of the first three coalitions in $\mathcal{B}$, each having a weight of $\frac{1}{3}$. The three other players are member of one two-player coalition with player 1 and of the three-player coalition 234. So, the balancing coefficients add up to unity for these three players as well. ■

We derive some basic properties of balanced collections and state them in the next theorem. The proof of this theorem is relegated to the appendix of this chapter.

Theorem 2.10 *Let N be some player set.*

(a) *The union of balanced collections on N is balanced.*
(b) *A balanced collection is minimal if and only if it has a unique set of balancing coefficients.*
(c) *Any balanced collection is the union of minimal balanced collections.* ■

Theorem 2.10(b) immediately implies the following characterization of minimal balanced collections.

Corollary 2.11 *A minimal balanced collection on player set N consists of at most $|N|$ coalitions.*

The proof is the subject of a problem in the problem section of this chapter and is, therefore, omitted.

The fundamental Bondareva–Shapley Theorem on the existence of Core allocations now can be stated as follows:

Theorem 2.12 (Bondareva–Shapley Theorem) *For any game* $v \in \mathcal{G}^N$ *we have that* $C(v) \neq \varnothing$ *if and only if for every balanced collection* $\mathcal{B} \subset 2^N$ *with balancing coefficients* $\{\lambda_S \mid S \in \mathcal{B}\}$ *it holds that*

$$\sum_{S \in \mathcal{B}} \lambda_S \, v(S) \leqq v(N) \tag{2.10}$$

For a proof of the Bondareva–Shapley Theorem I again refer to the appendix of this chapter.

From Theorem 2.10 the Bondareva–Shapley Theorem can be restated using only *minimal* balanced collections of coalitions rather than all balanced collections.

Corollary 2.13 *Let* $v \in \mathcal{G}^N$. *Then* $C(v) \neq \varnothing$ *if and only if for every minimal balanced collection* $\mathcal{B} \subset 2^N$ *with balancing coefficients* $\{\lambda_S \mid S \in \mathcal{B}\}$ *it holds that*

$$\sum_{S \in \mathcal{B}} \lambda_S \, v(S) \leqq v(N)$$

For three player games the Bondareva–Shapley Theorem can be stated in a very intuitive fashion:

Corollary 2.14 *Let* v *be a superadditive three player game. Then* $C(v) \neq \varnothing$ *if and only if*

$$v(12) + v(13) + v(23) \leqq 2\, v(123) \tag{2.11}$$

Proof As stated before $N = \{1, 2, 3\}$ has exactly one non-trivial minimal balanced collection, namely $\mathcal{B} = \{12, 13, 23\}$ with balancing coefficients $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The assertion now follows immediately from the application of Corollary 2.13 to this particular minimal balanced collection. ■

2.2 The Core Based on a Collection of Coalitions

Until now we worked under the fundamental hypothesis that *every* group of players is a coalition in the sense that it can establish collective, purposeful behavior. Hence, *every* group of players is assumed to have an internal structure that allows its members to cooperate with each other through interaction or communication and make collective decisions. Hence, the coalition can act like a "club" and has the necessary internal communication structure. Therefore, one might say that such a coalition has a "constitution"—an institutional collective conscience.[3] This seems a rather strong assumption.

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[3] Myerson (1980) refers to such institutionally endowed coalitions as "conferences". This seems less appropriate an indication since a conference usually refers to a formal meeting for discussion. In this regard the term of "institutional coalition" seems more to the point in an allocation process.

It is more plausible to assume that only a certain family of groups of players have a sufficient internal communication structure to operate as a coalition in cooperative game theoretic sense. Such a group of players can also be denoted as an *institutional coalition*. Formally, we introduce a collection $\Omega \subset 2^N$ of such institutional coalitions. Such a collection Ω is also called an *institutional coalitional structure* on N that describes certain institutional constraints on communication and cooperation between the various players in the game. Myerson (1980) was the first to consider the consequences of the introduction of such a limited class of institutional coalitions. He denoted the collection of institutional coalitions Ω as a "conference structure" on N.

Throughout we make the technical assumption that Ω is such that $\varnothing \in \Omega$. However, we do not assume that necessarily $N \in \Omega$, since the grand coalition N does not have to possess the appropriate structure to sustain internal communication, i.e., the grand coalition N does not have to be "institutional".

Example 2.15 Let N be an arbitrary player set. Then there are several well-investigated coalitional structures in the literature.

(a) If we choose Ω to be a partitioning of N, then Ω is called a *coalition structure*. In a seminal contribution Aumann and Drèze (1974) investigate the Core that results if only coalitions form within such a partitioning. I also refer to Greenberg (1994) for an elaborate discussion of the concept of coalition structure and its applications.

Usually these partitionings of the player set N are based on some social space or ordering. The coalitions that are feasible in such a social space are based on social neighborhoods, and are therefore indicated as *neighboring* coalitions. Consider $N = \{1, 2, 3, 4, 5\}$ where we place the players into a one-dimensional social space in the order of 1 through 5. Then an explicit example of a coalition structure is $\Omega = \{12, 3, 45\}$, where 12, 3 and 45 are the constituting neighboring coalitions.

Another example is given in a two-dimensional social space depicted in Fig. 2.5 for player set $N = \{1, 2, 3, 4, 5, 6, 7\}$. Here players are located in a two-dimensional social space and only neighboring players form institutional coalitions. In the given figure the neighborhoods are depicted as the red circles and the resulting coalition structure is $\Omega = \{123, 4, 5, 67\}$.

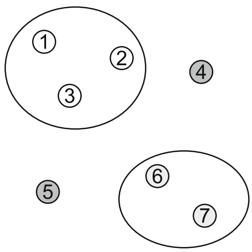

Fig. 2.5 Coalition structure of neighboring coalitions

(b) Greenberg and Weber (1986) introduced the notion of *consecutive* coalitions in the context of a public goods provision game. This refers to a specific type of neighboring coalitions based on an ordering of the players. In particular, consider the standard linear order of the player set $N = \{1, \dots, n\}$. Now a coalition $S \subset N$ is a consecutive coalition if $S = [i,j] = \{k \in N \mid i \leqq k \leqq j\}$ for some pair of players $i,j \in N$. Hence, in a consecutive coalition every "intermediate" player is also a member of that coalition. The collection of consecutive coalitions is now given by

$$\Omega^C = \{[i,j] \mid i < j,\ i,j \in N\}.$$

In particular, $\{i\} \in \Omega^C$ for every player $i \in N$.

(c) Myerson (1977) went one step further than the case of a coalition structure and considered cooperation under explicit communication restrictions imposed by a communication network among the players. Formally, we define a *(communication) network* on N by a set of communication links

$$g \subset \{ij \mid i,j \in N\} \tag{2.12}$$

where $ij = \{i,j\}$ is a binary set. If $ij \in g$, then it is assumed that players i and j are able to communicate and, thus, negotiate with each other.

Two players i and j are *connected* in the network g if these two players are connected by a path in the network, i.e., there exist $i_1, \dots, j_K \in N$ such that $i_1 = i$, $i_K = j$ and $i_k i_{k+1} \in g$ for all $k = 1, \dots, K-1$. Now a group of players $S \subset N$ is connected in the network g if all members $i,j \in S$ are connected in g. Thus, Myerson (1977) introduced

$$\Omega_g = \{S \subset N \mid S \text{ is connected in } g\} \tag{2.13}$$

In Myerson (1977) the main institutional feature of an institutional coalition is therefore that its members can communicate, either directly or indirectly, with each other. This rather weak requirement would suffice to establish some constitutional structure on the coalition.

Consider the communication network depicted in Fig. 2.6. The coalitions that can form in this communication network are exactly those that are connected in this network. Hence, 146 is a coalition since its members can communicate. However, 16 is not a formable coalition since players 1 and 6 require the assistance of player 4 to communicate.

In this chapter I further explore these cases in a limited fashion. In particular, I investigate the non-emptiness of the Core for these types of restrictions on coalition formation in the discussion of applications of the Kaneko–Wooders existence theorem.

We limit our investigations to the Core allocations that are generated within the context of a coalitional structure $\Omega \subset 2^N$.

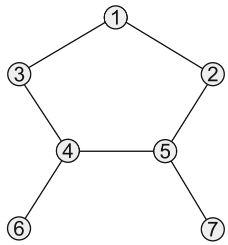

Fig. 2.6 A Myerson communication network

Definition 2.16 Let $\Omega \subset 2^N$. A vector $x \in \mathbb{R}^N$ is an Ω-*Core allocation* of the cooperative game $v \in \mathcal{G}^N$ if x satisfies the efficiency requirement

$$\sum_{i \in N} x_i = v(N) \tag{2.14}$$

and for every coalition $S \in \Omega$

$$\sum_{i \in S} x_i \geqq v(S). \tag{2.15}$$

Notation: $x \in C(\Omega, v)$.

We emphasize first that Ω-Core allocations are in general not satisfying the conditions of imputations. This is a consequence of the possibility that individual players might not be able to act independently from other players. This is particularly the case if permission is required for productive activities of such a player. Only if an individual player can disengage from such permission, she will be able to act independently.

I first investigate the geometric structure of the Ω-Core of a cooperative game for arbitrary collections of institutional coalitions $\Omega \subset 2^N$. In particular, it is clear that exploitation in Core allocations emerges if there are dependencies between different players in the game, i.e., if one player cannot act independently from another player as is the case in hierarchical authority situations.

In fact, only if the collection of institutional coalitions Ω is equal to the class of all possible groups of players 2^N it is guaranteed that the Core is in the imputation set. Indeed, $C(2^N, v) = C(v) \subset I(v)$. On the other hand if $\Omega \neq 2^N$, it is certainly not guaranteed that the resulting Core allocations are imputations. In particular, it has to hold that $\{i\} \in \Omega$ for every $i \in N$ in order that all allocations in $C(\Omega, v) \subset I(v)$.

Suppose that $x, y \in \mathbb{R}^N$ with $x \neq y$ are two allocations of $v(N)$. The difference $x - y \in \mathbb{R}^N$ is now obviously a transfer of side payments between the players in N. Such transfers are subject to the same considerations as Core allocations. Indeed, coalitions can block the execution of such transfers if they have the opportunity to provide its members with better side payments. For a class of institutional coalitions $\Omega \neq 2^N$ such stable side payments form a non-trivial class.

Definition 2.17 Let $\Omega \subset 2^N$. The set of *stable side payments* for Ω is defined by

$$\Gamma_\Omega = \left\{ x \in \mathbb{R}^N \;\middle|\; \begin{array}{l} \sum_N x_i = 0 \\ \sum_S x_i \geqq 0 \ \text{for}\, S \in \Omega \end{array} \right\} \tag{2.16}$$

We remark first that for any $\Omega \subset 2^N$ it holds that $\Gamma_\Omega \neq \varnothing$. Indeed, $0 \in \Gamma_\Omega$ for arbitrary Ω.

For the original situation that every group of players is formable as an institutional coalition, we arrive at a trivial collection of stable side payments. Indeed, $\Gamma_{2^N} = \{0\}$. This implies that *any* proposed transfer can be blocked. Hence, there is no justified exploitation of some players by other players. This is in fact a baseline case.

The other extreme case is $\Omega = \varnothing$. In that case it is easy to see that

$$\Gamma_\varnothing = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = 0 \right\} \sim \mathbb{R}^{n-1}.$$

The above states that the set of stable side payments for the empty class of institutional coalitions is isomorph to the $(n-1)$-dimensional Euclidean space, where $n = |N|$ is the number of players. This establishes that $\Gamma_\varnothing$ is the largest set of stable side payments. This can be interpreted as that the absence of any institutional coalitions allows for arbitrary exploitation of players by others. In this regard, $\Gamma_\varnothing$ reflects a complete absence of any organization of the players.

It is straightforward to see that if one adds a stable side payment vector to a Core allocation that one obtains again a Core allocation. A formal proof of the next proposition is therefore left as an exercise to the reader.

Proposition 2.18 *Let* $\Omega \subset 2^N$ *and* $v \in \mathcal{G}^N$. *Then*

$$C(\Omega, v) = C(\Omega, v) + \Gamma_\Omega \tag{2.17}$$

To illustrate these definitions I introduce a simple example.

Example 2.19 Let $N = \{1,2,3\}$ and let $\Omega = \{1, 12, 123\}$. Now the set of stable side payments is given by

$$\begin{aligned} \Gamma_\Omega &= \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0,\ x_1 + x_2 \geqq 0 \,\text{and}\, x_1 \geqq 0\} = \\ &= \{(\lambda, \mu - \lambda, -\mu) \mid \lambda, \mu \geqq 0\} = \\ &= \text{Cone}\,\{(1,-1,0),(0,1,-1)\}. \end{aligned}$$

From the properties of the collection Γ_Ω it follows immediately that player 1 has extraordinary power. She is able to get arbitrary transfers made to her from the other players. in this regard, player 1 has a *middleman position* in the interaction network among the three players. The only limitation is that player 2 has to be kept reasonably satisfied as well and should be treated better than player 3. However, player

1 can extract significant surpluses from her middleman position in the interaction network.

Another interpretation of the coalitional structure Ω is that the institutional coalitions are in fact generated through the implementation of an authority hierarchy: Player 1 is the top player, then follows player 2, and finally player 3 is at the bottom of this simple hierarchy. Hence, the middleman position of player 1 is equivalent to being the superior in an authority situation in which she exercises veto power over the actions considered by the other two players. ■

The discussion in these lecture notes will concentrate on the question under which conditions the set of stable side payments is the singleton $\{0\}$, describing the case of no exploitation. Subsequently I discuss the case of hierarchical authority structures and the resulting constraints on coalition formation and the structure of the Core.

I will not address the problem of the existence of restricted Core allocations. Faigle (1989) investigated the precise conditions on the collection of coalitions Ω and the cooperative game $v \in \mathcal{G}^N$ such that $C(\Omega, v)$ is non-empty.

A Reformulation Restricting the ability of certain groups of players to form a proper institutional coalition in the sense of a value-generating organized entity has consequences for what constitutes a Core allocation. If only certain groups of players can form as value-generating entities, then the generated values in the cooperative game should be based on these institutional coalitions only. This is exactly the line of reasoning pursued by Kaneko and Wooders (1982). They define a corresponding cooperative game that takes into account that only certain groups of players can form institutional coalitions.

Definition 2.20 Let $\Omega \subset 2^N$ be a collection of institutional coalitions such that $\{i\} \in \Omega$ for every $i \in N$.

(i) Let $S \subset N$ be an arbitrary coalition. A collection $P(S) \subset \Omega$ is called a Ω-*partition* of S if $P(S)$ is a partitioning of S consisting of institutional coalitions only. The family of all Ω-partitions of S is denoted by $\mathcal{P}_\Omega(S)$.
(ii) Let $v \in \mathcal{G}^N$ be an arbitrary cooperative game. The Ω-*partitioning game* corresponding to v is the cooperative game $v_\Omega \in \mathcal{G}^N$ given by

$$v_\Omega(S) = \max_{P(S)\in\mathcal{P}_\Omega(S)} \sum_{T\in P(S)} v(T) \tag{2.18}$$

for all coalitions $S \subset N$.

The following proposition is rather straightforward and, therefore, stated without a proof, which is left to the reader.

Proposition 2.21 (Kaneko and Wooders, 1982) *Let $\Omega \subset 2^N$ be a collection of institutional coalitions such that $\{i\} \in \Omega$ for every $i \in N$ and let $v \in \mathcal{G}^N$ be an arbitrary cooperative game. Then the Ω-Core of the game v is equivalent to the (regular) Core of the corresponding Ω-partitioning game:*

$$C(\Omega, v) = C(v_\Omega) .$$

For the standard case $\Omega = 2^N$ this translates into the following corollary:

Corollary 2.22 *The Core of an arbitrary cooperative game $v \in \mathcal{G}^N$ is equal to the Core of its* ***superadditive cover*** *$\hat{v} \in \mathcal{G}^N$ given by*

$$\hat{v}(S) = \max_{P_S \in \mathcal{P}(S)} \sum_{T \in P_S} v(T),$$

where $\mathcal{P}(S)$ is the collection of all finite partitions of the coalition $S \subset N$.

2.2.1 Balanced Collections

This section is based on Derks and Reijnierse (1998). They investigate the properties of the set of stable side payments Γ_Ω in case the collection of institutional coalitions Ω is non-degenerate and/or balanced. They show that balancedness again plays a central role with regard to exploitation within such coalitionally structured games.

A collection $\Omega \subset 2^N$ has a *span* given by all coalitions that can be generated by Ω. Formally,

$$\text{Span}(\Omega) = \left\{ S \subset N \;\middle|\; \chi_S = \sum_{R \in \Omega} \lambda_R \chi_R \text{ for some } \lambda_R \in \mathbb{R},\ R \in \Omega \right\} \tag{2.19}$$

where we recall that $\chi_S \in \{0, 1\}^N$ is the indicator vector of coalition $S \subset N$ with $\chi_S(i) = 1$ if and only if $i \in S$.

A collection of institutional coalitions $\Omega \subset 2^N$ is *non-degenerate* if $\text{Span}(\Omega) = 2^N$. This implies that the indicator vectors $\{\chi_R \mid R \in \Omega\}$ span the whole allocation space $\mathbb{R}^N$, i.e., these indicator vectors form a basis of $\mathbb{R}^N$.

Proposition 2.23 (Derks and Reijnierse, 1998, Theorem 4) *Let $\Omega \subset 2^N$. The set of stable side payments Γ_Ω is a pointed cone if and only if Ω is non-degenerate.*

Proof Remark that Γ_Ω is a pointed cone if and only if for any $x \in \mathbb{R}^N$ with $\sum_S x_i = 0$ for all $S \in \Omega$ we must have $x = 0$. This in turn is equivalent to the requirement that $\{\chi_S \mid S \in \Omega\}$ spans the Euclidean space $\mathbb{R}^N$. The latter is by definition the property that Ω is non-degenerate. ■

To illustrate this result and the definition of a non-degenerate collection of institutional coalitions we return to Example 2.19.

Example 2.24 Consider the Ω described in Example 2.19. We claim that this collection is non-degenerate. Indeed,

$$\{\chi_S \mid S \in \Omega\} = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\} ,$$

which is a basis for $\mathbb{R}^3$. This implies immediately that Ω is in fact spanning 2^N.

Also, it is immediately clear that Γ_Ω is a pointed cone. In fact we can write

$$\Gamma_\Omega = \text{Cone }\{(1,-1,0),(1,0,-1),(0,1,-1)\}$$

which lists the directional vectors of exploitative relationships within the hierarchy described in Example 2.19. Again it is clear from these descriptions that the middleman (Player 1) has power to exploit the situation to the detriment of the two other players 2 and 3. ■

We note that the non-degeneracy requirement is rather closely related to the balancedness condition of the collection Ω. For a balanced collection it is required that the balancing coefficients are strictly positive, while for non-degeneracy these coefficients can be arbitrary. A similar insight as Proposition 2.23 is therefore to be expected.

Proposition 2.25 (Derks and Reijnierse, 1998, Theorem 5) *Let $\Omega \subset 2^N$. The set of stable side payments Γ_Ω is a linear subspace of $\mathbb{R}^N$ if and only if Ω is balanced.*

Proof Suppose first that Ω is balanced with balancing coefficients λ. Thus, for short, $\chi_N = \sum_{S\in\Omega} \lambda_S \chi_S$.

Let $x \in \Gamma_\Omega$ and $S \in \Omega$. Then

$$-\sum_{i\in S} x_i = x \cdot \frac{1}{\lambda_S}\left(\sum_{T\in\Omega:\, T\neq S} \lambda_T \chi_T - \chi_N\right) = \frac{1}{\lambda_S}\sum_{T\in\Omega:\, T\neq S} \lambda_T \sum_{i\in T} x_i \geqq 0.$$

Since $S \in \Omega$ is arbitrary this shows that $-x \in \Gamma_\Omega$. Hence, Γ_Ω is a linear subspace of $\mathbb{R}^N$.

To show the converse, assume that Γ_Ω is a linear subspace of $\mathbb{R}^N$. The inequalities $\sum_N x_i \leqq 0$ and $\sum_S x_i \geqq 0$ for $S \in \Omega$ now imply that $\sum_T x_i \leqq 0$ for arbitrary $T \in \Omega$.

Using Farkas' Lemma (Rockafellar, 1970, Corollary 22.3.1) for linear spaces we can write

$$-\chi_T = \sum_{S\in\Omega} \mu_T(S)\chi_S - \nu_T \chi_N \tag{2.20}$$

for some well chosen coefficients $\mu_T \geqq 0$ and $\nu_T \geqq 0$. Adding (2.20) over all $T \in \Omega$ leads to

$$\nu\, \chi_N = \sum_{S\in\Omega}\left(1 + \sum_{T\in\Omega} \mu_T(S)\right) \chi_S \text{ with } \nu = \sum_{T\in\Omega} \nu_T.$$

Without loss of generality we may assume that $\nu > 0$. This in turn implies that Ω is in fact balanced. ■

From Propositions 2.23 and 2.25 we immediately conclude the following:

Corollary 2.26 *Let* $\Omega \subset 2^N$*. There is no exploitation, i.e.,* $\Gamma_\Omega = \{0\}$*, if and only if* Ω *is balanced as well as non-degenerate.*

In many ways this Corollary is the main conclusion from the analysis in this section. It provides a full characterization of the conditions under which there is no exploitation in stable side payments.

2.2.2 Strongly Balanced Collections

In the previous discussion I limited myself to the identification of some descriptive properties of an Ω-Core. In particular I identified the properties of the collection Ω of institutional coalitions that induce the resulting Ω-Core to be a linear space or a pointed cone. Thus far we did not discuss yet the non-emptiness of such Cores in the tradition of the Bondareva–Shapley Theorem 2.12.

The Core based on a collection Ω of institutional coalitions has the remarkable property that certain properties of the collection Ω imply that $C(\Omega, v) \neq \varnothing$ for *every* game $v \in \mathcal{G}^N$. This property is referred to as *strong balancedness* and was seminally explored in Kaneko and Wooders (1982). The main existence result of Kaneko and Wooders (1982) is therefore an extension of the Bondareva–Shapley Theorem 2.12.

I follow the formulation of the strong balancedness condition developed in Le Breton, Owen, and Weber (1992).

Definition 2.27 A collection $\Omega \subset 2^N$ of institutional coalitions is *strongly balanced* if $\{i\} \in \Omega$ for every player $i \in N$ and every balanced sub-collection $\mathcal{B} \subset \Omega$ contains a partitioning of N.

The remarkable extension of the Bondareva–Shapley Theorem can now be stated as follows. For a proof of this existence result, the Kaneko–Wooders theorem, we refer to the appendix of this chapter.

Theorem 2.28 (Kaneko and Wooders, 1982) *Let* $\Omega \subset 2^N$ *be some collection of institutional coalitions. For every cooperative game* $v \in \mathcal{G}^N$ *its* Ω*-Core is non-empty, i.e.,* $C(\Omega, v) \neq \varnothing$ *for all* $v \in \mathcal{G}^N$*, if and only if* $\Omega \subset 2^N$ *is strongly balanced.*

Le Breton et al. (1992) discuss the strong balancedness property in the context of the two well-known and -explored examples of consecutive coalitions and communication networks introduced in Example 2.15.

First, it is immediately clear that the collection of consecutive coalitions Ω^C is obviously strongly balanced. Indeed, the players are ordered using the standard number ordering and all consecutive coalitions based on this ordering are in the collection Ω^C.

Second, in Myerson's communication network model one can ask the question when the collection of connected coalitions is strongly balanced. For the discussion of this we introduce some auxiliary concepts from graph or network theory.

Let $g \subset \{ij \mid i,j \in N,\ i \neq j\}$ be some network on the player set N, where $ij = \{i,j\}$ is an unordered pair of players. A *path* in the network g is a sequence of players $i_1, \ldots, i_K \in N$ such that $i_k i_{k+1} \in g$ for every $k \in \{1, \ldots, K-1\}$. A path consisting of the players $i_1, \ldots, i_K \in N$ in g is a *cycle* if $K \geqq 3$ and $i_1 = i_K$. Thus, a cycle is a communication path from a player to herself that consists of more than two players. Now a network g on N is *acyclic* if it does not contain any cycles.

Using the notion of an acyclic network we now arrive at the following characterization of strongly balanced collections in the Myerson network model.

Proposition 2.29 (Le Breton, Owen, and Weber, 1992) *Let g be a communication network on the player set N and let $\Omega_g \subset 2^N$ be the corresponding collection of connected coalitions in g. Then the collection Ω_g is strongly balanced if and only if the communication network g is acyclic.*

For a proof of this proposition I refer to the seminal source, Le Breton et al. (1992, pp. 421–422).

This main insight from Le Breton et al. (1992) shows that cycles in networks have a very important role in the allocation of values among the players. Communication cycles usually result in ambiguities about how players are connected and offer players multiple paths to link to other players to form coalitions. This results into auxiliary blocking opportunities, thus causing the Core to be possibly empty. A similar insight was derived by Demange (1994) for a slightly more general setting.

2.2.3 Lattices and Hierarchies

We continue with the discussion of a very common economic and social phenomenon, namely the consequences of the exercise of authority in hierarchically structured organizations.[4] We limit ourselves here to the study of the consequences of authority on coalition formation and the resulting Core.

The seminal contribution to cooperative game theory introducing hierarchical authority constraints on coalition formation was made by Gilles, Owen, and van den Brink (1992), who discussed so-called conjunctive authority structures. The next definition presents the formal introduction of such authority structures.

Definition 2.30 A map $H \colon N \to 2^N$ is a *permission structure* on player set N if it is irreflexive, i.e., $i \notin H(i)$ for all $i \in N$.

A permission structure $H \colon N \to 2^N$ is *strict* if it is acyclic, i.e., there is no sequence of players $i_1, \ldots, i_K \in N$ such that $i_{k+1} \in H(i_k)$ for all $k \in \{1, \ldots, K-1\}$ and $i_1 = i_K$.

The players $j \in H(i)$ are called the *subordinates* of player i in H. Similarly, the players $j \in H^{-1}(i) = \{j \in N \mid i \in S(j)\}$ are called player i's *superiors* in H. In certain permission structures all players might have subordinates as well as superiors.

[4] Such hierarchically structured organizations are very common in our society. Most production activities in the capitalist society are organized in hierarchical firms. Authority of superior over subordinates is very common and well accepted. The study of these organization structures has resulted into a very sizeable literature.

However, in a strict permission structure there are players that have no superiors and there are players that have no subordinates. In that respect a strict permission structure represents a true hierarchy.

Coalition formation in permission structures can be described in various different ways. The most straightforward description is developed in Gilles et al. (1992). We extend the notation slightly and denote by $H(S) = \cup_{i\in S} H(i)$ the subordinates of the coalition $S \subset N$.

Definition 2.31 Let H be a permission structure on N. A coalition $S \subset N$ is *conjunctively autonomous* in H if $S \cap H(N \setminus S) = \varnothing$. The collection of conjunctively autonomous coalitions is denoted by

$$\Omega_H = \{S \subset N \mid S \cap H(N \setminus S) = \varnothing\}.$$

Under the conjunctive approach[5] a coalition is autonomous if it does not have a member that has a superior outside the coalition. Hence, an autonomous coalition is hierarchically "self-contained" in the sense that all superiors of its members are members as well.

It is clear that $\Omega_H \subset 2^N$ is a rather plausible description of institutional coalitions under an authority structure H. Indeed, a coalition has to be autonomous under the authority structure H to be acting independently. It is our goal in this section to study the Core of such a collection of conjunctively autonomous coalitions.

We will first show that the introduction of a conjunctive authority structure on coalition formation is equivalent to putting lattice constraints on the collection of institutional coalitions. The material presented here is taken from Derks and Gilles (1995), who address the properties a these particular collections of coalitions.

Definition 2.32 A collection of coalitions $\Omega \subset 2^N$ is a *lattice* if $\varnothing, N \in \Omega$ and $S \cap T$, $S \cup T \in \Omega$ for all $S, T \in \Omega$.

A lattice is a collection that is closed for taking unions and intersections. Since N is finite, it therefore is equivalent to a topology on N.

We introduce some auxiliary notation. For every $i \in N$ we let

$$\partial_i\Omega = \cap\{S \in \Omega \mid i \in S\} \in \Omega \tag{2.21}$$

The map ∂_i simply assigns to player i the smallest institutional coalition that player i is a member of. We observe that $\partial_j\Omega \subset \partial_i\Omega$ if $j \in \partial_i\Omega$. Finally, we remark that

$$\partial\Omega = \{\partial_i\Omega \mid i \in N\}$$

[5] The term "conjunctive" refers to the veto power that can be exercised within the hierarchy introduced. Indeed each superior is in fact assumed to have full veto power over her subordinates. Only if all superiors of a player sign off in unison—or conjunctively—this player is allowed to go ahead with her planned actions.

is a basis of the lattice Ω, i.e., every coalition in Ω can be written as a union of $\partial\Omega$-elements. Also, $\partial\Omega$ is the unique minimal—or "smallest"—basis of the lattice Ω.

The link of lattices of institutional coalitions is made in the following result. For a proof I refer to the appendix of this chapter.

Theorem 2.33 *Let $\Omega \subset 2^N$ be any collection of coalitions. The following statements are equivalent:*

(i) *Ω is a lattice.*
(ii) *There is a permission structure H on N such that $\Omega = \Omega_H$.*

It is rather remarkable that hierarchical authority structures generate lattices and that each lattice corresponds to some authority structure. For such situations we can provide a complete characterization of the set of stable side payments. The proof of the next theorem is relegated to the appendix of this chapter.

Theorem 2.34 (Derks and Gilles, 1995, Theorem 2.3) *If $\Omega \subset 2^N$ is a lattice on N, then*

$$\Gamma_\Omega = \text{Cone}\,\left\{ e_j - e_i \mid i \in N \,\text{and}\, j \in \partial_i\Omega \right\} \tag{2.22}$$

where $e_h = \chi_{\{h\}} \in \mathbb{R}^N_+$ is the h-th unit vector in $\mathbb{R}^N$, $h \in N$.

Next we restrict our attention to a special class of lattices or authority structures. This concerns the class of lattices that is generated by strictly hierarchical authority structures. The main property that characterizes this particular class is that each individual player can be fully identified or determined through the family of coalition that she is a member of.

Definition 2.35 A collection of coalitions $\Omega \subset 2^N$ on N is *discerning* if for all $i, j \in N$ there exists some coalition $S \in \Omega$ such that either $i \in S$ and $j \notin S$ or $i \notin S$ and $j \in S$.

The properties that a discerning lattice satisfies is listed in the following theorem.[6] For a proof we again refer to the appendix of this chapter.

Theorem 2.36 *Let $\Omega \subset 2^N$ be a lattice on N. Then the following statements are equivalent:*

(i) *Ω is a discerning lattice.*
(ii) *For all $i, j \in N$: $\partial_i\Omega \neq \partial_j\Omega$.*
(iii) *For every $S \in \Omega \setminus \{\varnothing\}$ there exists a player $i \in S$ with $S \setminus \{i\} \in \Omega$.*
(iv) *There exists a strict permission structure H such that $\Omega = \Omega_H$.*

---

[6] A property that is not listed in this theorem is that a discerning lattice on N corresponds to a topology that satisfies the T_0-separation property.

(v) *For every player* $i \in N$: $\partial_i\Omega \setminus \{i\} \in \Omega$.
(vi) Ω *is a non-degenerate lattice.*

From this list of properties the following two conclusions can be drawn:

Corollary 2.37

(a) *The set of stable side payments* Γ_Ω *for a lattice* Ω *is a pointed cone if and only if* Ω *is discerning.*
(b) *The set of all coalitions* 2^N *is the only lattice* Ω *for which* $\Gamma_\Omega = \{0\}$.

We further explore some properties of the Core under lattice constraints on coalition formation in the next section on so-called Core catchers or Core covers.

2.3 Core Covers and Convex Games

We complete our discussion of the Core of a cooperative game by reviewing some supersets of the Core, also called “Core covers” or “Core catchers”. Formally, a *Core cover* is a map $\mathcal{K}\colon \mathcal{G}^N \to 2^{\mathbb{R}^N}$ such that $C(v) \subset \mathcal{K}(v)$. I limit the discussion to two recently introduced and well known Core cover concepts, the Weber set (Weber, 1988) and the Selectope (Derks, Haller, and Peters, 2000).

The *Weber set* is founded on the principle that players enter the cooperative game in random order and collect their marginal contribution upon entering. The random order of entry is also called a “Weber string” and the generated payoff vector is called the corresponding “marginal (payoff) vector”. Arbitrary convex combinations of such marginal vectors now represent players entering the game and collecting payoff with arbitrary probability distributions. Hence, the Weber set—defined as the convex hull of the set of these marginal vectors—is now the set of all these expected payoffs. In general the Core is a subset of the Weber set.

The *Selectope* is based on a closely related principle. Coalitions are assigned their Harsanyi dividend in the cooperative game that they participate in. A single member of each coalition is now selected to collect this dividend. These selections are in principle arbitrary. Now the Selectope—being the convex hull of all corresponding payoff vectors to such selected players—can be shown to be usually a superset of the Weber set.

2.3.1 The Weber Set

Throughout this section we develop the theory of the Weber set for a *discerning lattice* of coalitions $\Omega \subset 2^N$.[7] The main results regarding the Weber set and its rela-

[7] Equivalently, we may assume that the player set N is endowed with an acyclic or strict permission structure H with $\Omega_H = \Omega$. Throughout this section I prefer to develop the theory of the Weber set for the concept of a discerning lattice.

tion to the Core are developed for general discerning lattices of feasible institutional coalitions. We state results for the regular Core as corollaries of these main theorems for $\Omega = 2^N$.

Definition 2.38 Let $v \in \mathcal{G}^N$.

(i) A *Weber string* in Ω is a permutation $\rho\colon N \rightleftarrows N$ such that $\{\rho(1), \dots, \rho(k)\} \in \Omega$ for every $k \in N$.
(ii) The *Weber set* for v is given by

$$\mathcal{W}(\Omega, v) = Conv\,\{x^\rho \in \mathbb{R}^N \mid \rho \text{ is a Weber string in } \omega\}, \tag{2.23}$$

where $Conv\,(X)$ is the convex hull[8] of $X \subset \mathbb{R}^N$ and for every Weber string ρ in Ω we define

$$x_i^\rho = v(R_i) - v(R_i \setminus \{i\}) \tag{2.24}$$

with $R_i = \{\rho(1), \dots, \rho(j)\}$, where $j \in N$ is such that $\rho(j) = i$.

The existence of Weber strings have to be established in order to guarantee the non-emptiness of the Weber set.

Lemma 2.39

(a) *For a lattice Ω it holds that there exists a Weber string in Ω if and only if Ω is discerning.*
(b) *Every permutation is a Weber string in 2^N implying that the regular Weber set is given by*

$$\mathcal{W}(v) = \mathcal{W}(2^N, v) = \text{Conv}\,\{x^\rho \mid \rho \textit{ is a permutation on } N\}.$$

The proof of Lemma 2.39(a) is relegated to the problem section of this chapter. Lemma 2.39(b) is straightforward.

The Weber set $\mathcal{W}(\Omega, v)$ on a discerning lattice Ω has some appealing properties. Let H be an acyclic permission structure such that $\Omega = \Omega_H$. Let $S \in \Omega$ and consider the corresponding unanimity game $u_S \in \mathcal{G}^N$. Then

$$\mathcal{W}(\Omega, u_S) = \text{Conv}\,\{e_i \mid i \in D_H(S)\} \text{ with} D_H(S) = \{i \in S \mid H(i) \cap S = \varnothing\}.$$

[8] The convex hull of a set X is defined as

$$\text{Conv}\,(X) = \left\{ \sum_{k=1}^{M} \lambda_k \cdot x_k \;\middle|\; x_1, \dots, x_m \in X \text{ and} \sum_{k=1}^{m} \lambda_k = 1 \right\}.$$

This means that in the Weber set of a unanimity game the coalition's wealth is fully distributed over the players in the lowest level of the hierarchical structure H restricted to the coalition S.

The main theorem that determines the relationship of the Core with the Weber set is stated for convex games. The next definition generalizes the convexity notion seminally introduced by Shapley (1971).

Definition 2.40 A game $v \in \mathcal{G}^N$ is *convex* on the discerning lattice Ω if for all $S, T \in \Omega$:

$$v(S) + v(T) \leqq v(S \cup T) + v(S \cap T). \tag{2.25}$$

The main theorem can now be stated. For a proof I refer to the appendix of this chapter.

Theorem 2.41 (Derks and Gilles, 1995) *Let $\Omega \subset 2^N$ be a discerning lattice and $v \in \mathcal{G}^N$.*

(a) $C(\Omega, v) \subset \mathcal{W}(\Omega, v) + \Gamma_\Omega$.
(b) *The game v is convex on Ω if and only if*

$$C(\Omega, v) = \mathcal{W}(\Omega, v) + \Gamma_\Omega.$$

From the fact that $\Gamma_{2^N} = \{0\}$ the original seminal results from Shapley (1971) and Ichiishi (1981) can be recovered from Theorem 2.41.

Corollary 2.42 *Let $v \in \mathcal{G}^N$. Then*

(a) $C(v) \subset \mathcal{W}(v)$ *and*
(b) $C(v) = \mathcal{W}(v)$ *if and only if v is convex on 2^N.*

2.3.2 The Selectope

The Selectope has seminally been introduced by Hammer, Peled, and Sorensen (1977) and was further developed by Derks, Haller, and Peters (2000). In this section I will develop the theory of the Selectope through the analysis presented in Derks et al. (2000). Throughout this section I refer for detailed proofs of the main propositions to that paper. It would demand too much to develop these elaborate proofs here in full detail. In the previous section we developed the theory of the Weber set for discerning lattices of coalitions; for the Selectope we return to the standard case considering all feasible coalitions, i.e., $\Omega = 2^N$.

The Selectope is a cousin of the Weber set in the sense that both concepts consist of convex combinations of marginal value allocations. In the Weber set these marginal values are based on Weber strings. In the Selectope these marginal values are based on selectors.

Definition 2.43 Let $v \in \mathcal{G}^N$.

A *selector* is a function $\alpha\colon 2^N \to N$ with $\alpha(S) \in S$ for all $S \neq \varnothing$.

The *selector allocation* corresponding to the selector α is $m^\alpha \in \mathbb{R}^N$ defined by

$$m_i^\alpha(v) = \sum_{S\colon i=\alpha(S)} \Delta_v(S), \tag{2.26}$$

where $\Delta_v(S)$ is the Harsanyi dividend of S in the game v.

The *Selectope* of the game v is now defined as

$$\mathcal{S}(v) = Conv\left\{ m^\alpha(v) \in \mathbb{R}^N \;\middle|\; \alpha\colon 2^N \to N \text{ is a selector} \right\} \tag{2.27}$$

The Selectope of a game is the collection of all reasonable allocations based on the coalitional dividends that are generated in this game. Our goal is to provide a characterization of the Selectope of any game. For that we have to develop some more notions.

Recall that we can decompose a game v using the unanimity basis by $s = \sum_{S \subset N} \Delta_v(S) u_S$. Now we introduce a positive and negative part of the game v by

$$v^+ = \sum_{S\colon \Delta_v(S)>0} \Delta_v(S)\, u_S$$
$$v^- = \sum_{S\colon \Delta_v(S)<0} -\Delta_v(S)\, u_S$$

It immediately follows that $v = v^+ - v^-$. We also mention here that a game is called *positive* if $v = v^+$, *negative* if $v = -v^-$, and *almost positive* if $\Delta_v(S) \geqq 0$ for all $S \in 2^N$ with $|S| \geqq 2$.[9]

Definition 2.44 The *dual* of a game $v \in \mathcal{G}^N$ is defined by the game $v^* \in \mathcal{G}^N$ with

$$v^*(S) = v(N) - v(N \setminus S) \text{ for all } S \subset N.$$

The following proposition combines Lemma 3 and Theorems 1 and 2 of Derks et al. (2000). For a proof of the statements in this proposition I refer to that paper.

Proposition 2.45 *Let* $v \in \mathcal{G}^N$.

(a) $\mathcal{S}(v) = C(v^+) - C(v^-)$.
(b) *The Selectope* $\mathcal{S}(v)$ *of* v *is equal to the Core of the convex game* $\tilde{v} = v^+ + \left(-v^-\right)^*$*, which in turn implies that* $\mathcal{S}(v) = \mathcal{W}(\tilde{v})$.
(c) *The following statements are equivalent:*

---

[9] A game is therefore almost positive if it can be written as the sum $v = v^+ + \sum_{i \in N^-} v(\{i\})\, u_i$, where $N^- = \{i \in N \mid v(\{i\}) < 0\}$.

(i) *$C(v) = \mathcal{S}(v)$;*
(ii) *$\mathcal{S}(v) \subset I(v)$, and*
(iii) *v is almost positive.*

Furthermore, $\mathcal{S}(v) = I(v)$ if and only if v is additive.

Since the Selectope is usually strictly larger than the Weber set as well as the Core of a game, it is a concept that is of less interest than the Core of a game. In that regard the Selectope is indeed a true Core "catcher".

This completes our discussion of the Selectope of a game. For further analysis of the Selectope I refer to Derks et al. (2000).

2.4 Appendix: Proofs of the Main Theorems

Proof of Theorem 2.5

Proof of (a)
Let $x \in C(v) \subset I(v)$ and suppose that there exists some $y \in I(v)$ that dominates x through the coalition S. Then it has to hold that $x_i < y_i$ for every player $i \in S$ and, additionally, that $\sum_{i \in S} y_i \leqq v(S)$. Therefore,

$$\sum_{i \in S} x_i < \sum_{i \in S} y_i \leqq v(S),$$

contradicting the Core requirement that $\sum_{i \in S} x_i \geqq v(S)$.

Proof of (b)
Let v be superadditive. Since assertion (a) has been shown above, it only remains to be proven that every undominated imputation is in the Core of v.

Let $x \in I(v)$ be such that $x \notin C(v)$. Then it has to follow that $\sum_N x_i \neq v(N)$ and/or $\sum_S x_i > v(S)$ for some coalition $S \subset N$.

By definition of an imputation the first is impossible; therefore, the latter holds, i.e., $\sum_S x_i > v(S)$ for some coalition $S \subset N$.

Now let $\varepsilon = v(S) - \sum_{i \in S} x_i > 0$. Also define

$$\alpha = v(N) - v(S) - \sum_{j \in N \setminus S} v(\{j\}).$$

Then by superadditivity it holds that $\alpha \geqq 0$.

Finally, we introduce the following imputation:

$$y_j = \begin{cases} x_j + \frac{\varepsilon}{|S|} & \text{if} j \in S \\ v(\{j\}) + \frac{\alpha}{n-|S|} & \text{if} j \notin S \end{cases}$$

First, we remark that $y_j \geqq v(\{j\})$ for all $j \in N$, since $x_i \geqq v(\{i\})$ for all $i \in S$ and the definition of y given above. Furthermore, it follows from the definition of y that

$$\sum_{j \in N} y_j = \sum_{S} x_i + \varepsilon + \sum_{N \setminus S} v(\{j\}) + \alpha =$$
$$= \sum_{S} x_i + \varepsilon + v(N) - v(S) = v(N).$$

Hence, we conclude that $y \in I(v)$. Moreover, from the definition we have that $y_i > x_i$ for all $i \in S$. Together this in turn implies that y dominates x through coalition S. This is a contradiction.

Proof of Theorem 2.10

This proof is based on the structure presented in Owen (1995, pp. 226–228).

Let N be some player set. We first show an intermediate result that is necessary for the proof of the three assertions stated in Theorem 2.10.

Lemma 2.46 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two balanced collections on N such that $\mathcal{B}_1 \subsetneq \mathcal{B}_2$. Then there exists a balanced collection $\mathcal{C}$ on N such that $\mathcal{B}_1 \cup \mathcal{C} = \mathcal{B}_2$ with $\mathcal{C} \neq \mathcal{B}_2$. Furthermore, the balancing coefficients for $\mathcal{B}_2$ are not unique.*

Proof Without loss of generality we may write

$$\mathcal{B}_1 = \{S_1, \ldots, S_k\}$$
$$\mathcal{B}_2 = \{S_1, \ldots, S_k, S_{k+1}, \ldots, S_m\}$$

with balancing coefficients $(\lambda_1, \ldots, \lambda_k)$ and $(\mu_1, \ldots, \mu_m)$ respectively.

Let $t > 0$ be any number. Now define

$$\nu_h = \begin{cases} (1+t)\mu_h - t\lambda_h & \text{if } h \in \{1, \ldots, k\} \\ (1+t)\mu_h & \text{if } h \in \{k+1, \ldots, m\} \end{cases}$$

For small enough t it is clear that $\nu_h > 0$ for all $h = 1, \ldots, m$. In that case it also holds for every player $i \in N$ that

$$\sum_{h:\, i \in S_h \in \mathcal{B}_2} \nu_h = (1+t) \sum_{h:\, i \in S_h \in \mathcal{B}_2} \mu_h - t \sum_{h:\, i \in S_h \in \mathcal{B}_1} \lambda_h = (1+t) - t = 1,$$

implying that ν is a vector of balancing coefficients for $\mathcal{B}_2$. It follows that ν is not unique.

Now, there must be at least one $j \in \{1, \ldots, k\}$ with $\lambda_j > \mu_j$. Indeed, otherwise for player $i \in S_{k+1}$ it has to hold that

$$1 = \sum_{h\colon i\in S_h\in\mathcal{B}_1} \lambda_h \leqq \sum_{h\colon i\in S_h\in\mathcal{B}_1} \mu_h < \sum_{h\colon i\in S_h\in\mathcal{B}_2} \mu_h = 1.$$

This would be a contradiction.

Now let

$$\hat{t} = \min\left\{ \frac{\mu_h}{\lambda_h - \mu_h} \,\middle|\, \lambda_h > \mu_h \right\}.$$

We introduce the collection

$$\mathcal{C}' = \{S_h \mid S_h \in \mathcal{B}_1,\ (1+\hat{t})\mu_h = \hat{t}\lambda_h\} \neq \varnothing$$

and let $\mathcal{C} = \mathcal{B}_2 \backslash \mathcal{C}'$. Obviously this collection satisfies the requirements formulated in the lemma. Finally, we complete the proof by showing that the defined coefficients ν for $\hat{t}$ is a set of balancing coefficients for the collection $\mathcal{C}$.

Indeed, we can write for any $i \in N$:

$$\begin{aligned}
\sum_{S\in\mathcal{C}\colon i\in S} \nu_S &= \sum_{S\in\mathcal{C}\colon i\in S} \left[(1+\hat{t})\mu_S - \hat{t}\lambda_S\right] \\
&= (1+\hat{t}) \sum_{S\in\mathcal{C}\colon i\in S} \mu_S - \hat{t} \sum_{S\in\mathcal{C}\colon i\in S} \lambda_S \\
&= (1+\hat{t}) \left(1 - \sum_{S\in\mathcal{C}'\colon i\in S} \mu_S\right) - \hat{t}\left(1 - \sum_{S\in\mathcal{C}'\colon i\in S} \lambda_S\right) \\
&= 1 - \sum_{S\in\mathcal{C}'\colon i\in S} \left[(1+\hat{t})\mu_S - \hat{t}\lambda_S\right]
\end{aligned}$$

Now for every coalition $S \in \mathcal{C}'$ we have that

$$\frac{\mu_S}{\lambda_S - \mu_S} = \hat{t},$$

and, therefore, for every $S \in \mathcal{C}'$:

$$(1+\hat{t})\mu_S - \hat{t}\lambda_S = \mu_S + \hat{t}(\mu_s - \lambda_S) = 0.$$

Together with the above this implies indeed that $\sum_{S\in\mathcal{C}\colon i\in S} \nu_S = 1$ for every $i \in N$. This in turn completes the proof that $\mathcal{C}$ is a balanced collection. ■

I now turn to the proof of the assertions stated in Theorem 2.10 using the formulated lemma.

Proof of (a)
We let $\mathcal{B}_1 = \{S_1, \dots, S_k\}$ and $\mathcal{B}_2 = \{T_1, \dots, T_m\}$ be two balanced collections on N with balancing coefficients $(\lambda_1, \dots, \lambda_k)$ and $(\mu_1, \dots, \mu_m)$ respectively. Now

$$\mathcal{C} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{U_1, \ldots, U_p\}$$

where $p \leqq k + m$. Let $0 < t < 1$. Define

$$\nu_j = \begin{cases} t\lambda_i & \text{if } U_j = S_i \in \mathcal{B}_1 \setminus \mathcal{B}_2 \\ (1-t)\mu_h & \text{if } U_j = T_h \in \mathcal{B}_2 \setminus \mathcal{B}_1 \\ t\lambda_i + (1-t)\mu_h & \text{if } U_j = S_i = T_h \in \mathcal{B}_1 \cap \mathcal{B}_2 \end{cases}$$

As one can verify, $(\nu_1, \ldots, \nu_p)$ are balancing coefficients for the collection $\mathcal{C}$. Hence, the union of two balanced collections is indeed balanced.

Now by induction on the number of collections, the union of any family of balanced collections has to be balanced as well. This completes the proof of (a).

Proof of (b)
Lemma 2.46 implies that the balancing coefficients will only be unique for minimally balanced collections. This leaves the converse to be shown.

Let $\mathcal{B} = \{S_1, \ldots, S_m\}$. Suppose that $\mathcal{B}$ has two distinct sets of balancing coefficients, λ and μ with $\lambda \neq \mu$. We may assume that $\lambda_j > \mu_j$ for at least one $j \in \{1, \ldots, m\}$. Again define $\nu = (1 + \hat{t})\,\mu - \hat{t}\,\lambda$ where

$$\hat{t} = \min \left\{ \frac{\mu_h}{\lambda_h - \mu_h} \;\middle|\; \lambda_h > \mu_h \right\}.$$

Now—following the same construction as in the proof of Lemma 2.46—ν is a set of balancing coefficients for the collection

$$\mathcal{C} = \{S_h \mid (1 + \hat{t})\,\mu_h \neq \hat{t}\,\lambda_h\}.$$

Since $\mathcal{C} \subsetneq \mathcal{B}$ we have shown that $\mathcal{B}$ is not minimal.

Proof of (c)
Let $\mathcal{B}$ be some balanced collection on N. We prove Assertion (c) by induction on the number B of coalitions in the collection $\mathcal{B}$.

For $B = 1$ there is only one balanced collection, namely $\mathcal{B} = \{N\}$. This is clearly a minimal balanced collection.

Next suppose the assertion is true for $B - 1$ and lower. Suppose that $|\mathcal{B}| = B$.

If $\mathcal{B}$ is minimal itself, then the assertion is trivially satisfied. If $\mathcal{B}$ is not minimal, then it has a proper subcollection $\mathcal{C} \subsetneq \mathcal{B}$ that is balanced. By Lemma 2.46 there exists another proper subcollection $\mathcal{D} \subsetneq \mathcal{B}$ such that $\mathcal{C} \cup \mathcal{D} = \mathcal{B}$. Since $\mathcal{C}$ and $\mathcal{D}$ are proper subcollections, they each have $B - 1$ or fewer coalitions. By the induction hypothesis each can be expressed as the union of minimally balanced subcollections. Hence, $\mathcal{B}$ itself is the union of minimally balanced subcollections.

Proof of Theorem 2.12

The proof of the Bondareva–Shapley Theorem is based on duality theory in linear programming. Let $v \in \mathcal{G}^N$. First, it has to be observed that $C(v) \neq \varnothing$ if and only if the following linear programming problem has a solution:

$$\text{Minimize} \quad P = \textstyle\sum_{i=1}^{n} x_i \tag{2.28}$$

$$\text{subject to} \quad \textstyle\sum_{i \in S} x_i \geqq v(S) \text{ for all } S \subset N \tag{2.29}$$

We now claim that $x \in C(v)$ if and only if it is a solution to this linear program such that $P = \sum_i x_i \leqq v(N)$.

First observe that indeed a solution $x^\star \in \mathbb{R}^N$ of this linear program such that the generated optimum $P^\star = \sum_{i=1}^{n} x_i^\star \leqq v(N)$ is indeed a Core imputation of v.

Conversely, if $x \in C(v)$, then it obviously satisfies the conditions for the linear program.

Hence, $\sum_i x_i = v(N)$ implying that $P = \sum_i x_i = v(N)$ is indeed a solution to the stated linear program.

Next we construct the dual linear program of the one formulated above. This is the linear program described by

$$\text{Maximize} \quad Q = \textstyle\sum_{S \subset N} y_s \, v(S) \tag{2.30}$$

$$\text{subject to} \quad \textstyle\sum_{S:\, i \in S} y_S = 1 \qquad \text{for all } i \in N \tag{2.31}$$

$$y_S \geqq 0 \qquad \text{for all } S \subset N \tag{2.32}$$

Both linear programs as stated, are feasible. Hence, the minimum $P^\star$ has to be equal to the maximum $Q^\star$. This in turn implies that $C(v) \neq \varnothing$ if and only if $Q^\star = P^\star \leqq v(N)$.

But the stated maximization problem exactly requires the identification of balanced collections of coalitions over which is maximized. A simple reformulation immediately leads to the stated conditions in the Bondareva–Shapley Theorem 2.12.

Proof of Theorem 2.28

Only if: Let $\Omega \subset 2^N$ be a collection that is not strongly balanced. Then there exists a sub-collection $\Gamma \subset \Omega$ that is minimally balanced and is *not* a partitioning of N. (The latter statement is equivalent to the negation of the definition of strong balancedness.)

For Γ we now define the game $w \in \mathcal{G}^N$ given by

$$w(S) = \begin{cases} |S| & \text{if } S \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

We show that $C(w,\Omega) = C(w_\Omega) = \varnothing$ by application of the Bondareva–Shapley Theorem.

Let the balancing coefficients of Γ be given by $\{\lambda_S \mid S \in \Gamma\}$. Then

$$\sum_{S\in\Gamma} \lambda_S w_\Omega(S) = \sum_{S\in\Gamma} \lambda_S w(S) = \sum_{i\in N} \sum_{S\in\Gamma:\, i\in S} \lambda_S = |N| = n.$$

But since Γ does not contain a partititioning of N, we arrive at

$$w_\Omega(N) = \max_{P(N)\in\mathcal{P}_\Omega(N)} \sum_{T\in P(N)} w(T) < |N| = n.$$

Thus, with the above it follows that

$$\sum_{S\in\Gamma} \lambda_S w_\Omega(S) = n > w_\Omega(N).$$

From the Bondareva–Shapley Theorem it thus follows that $C(w_\Omega) = \varnothing$, which implies the desired assertion.

If: Let $\Omega \subset 2^N$ be a strongly balanced collection. Consider an arbitrary game $v \in \mathcal{G}^N$. We now check the Bondareva–Shapley conditions for v_Ω to show that $C(v,\Omega) = C(v_\Omega) \neq \varnothing$.

(A) First, let $\Gamma \subset \Omega$ be some minimally balanced collection. Since, by assumption, Γ contains a partitioning P^* of N, this implies that $\Gamma = P^*$. Thus, the balancing coefficients for Γ are unique and given by $\lambda_S = 1$ for all $S \in \Gamma$. Furthermore, since $S \in \Gamma \subset \Omega$, it follows immediately that $v_\Omega(S) = v(S)$. This implies that

$$\sum_{S\in\Gamma} \lambda_S v_\Omega(S) = \sum_{T\in P^*} v(T) \leqq \max_{P\in\mathcal{P}_\Omega(N)} \sum_{T\in P} v(T) = v_\Omega(N)$$

Hence, the Bondareva–Shapley condition is satisfied for any minimally balanced sub-collection of Ω.

(B) Next, let Γ be a minimally balanced collection with $\Gamma \setminus \Omega \neq \varnothing$. Let $\{\lambda_S \mid S \in \Gamma\}$ be the balancing coefficients for Γ.

Now, by definition, for every $S \in \Gamma$ with $S \notin \Omega$ there exists some Ω-partitioning $P^*(S) \in \mathcal{P}_\Omega(S)$ such that $v_\Omega(S) = \sum_{T\in P^*(S)} v(T)$. For every $T \in \Omega$ we now define

$$\Gamma_T = \{S \in \Gamma \mid S \notin \Omega \,\text{and} T \in P^*(S)\}.$$

We now introduce a modified collection $\widehat{\Gamma} \subset \Omega$ given by

$$\widehat{\Gamma} = \{T \mid T \in \Gamma \cap \Omega\} \cup \left(\bigcup_{S\in\Gamma:\, S\notin\Omega} P^*(S) \right).$$

We also introduce modified balancing coefficients for the modified collection $\widehat{\Gamma}$ given by

$$\delta_T = \begin{cases} \lambda_T + \sum_{S \in \Gamma_T} \lambda_S & \text{if } T \in \Gamma \cap \Omega \\ \sum_{S \in \Gamma_T} \lambda_S & \text{if } T \notin \Omega \text{ and } T \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

It can now be verified rather easily that $\{\delta_T \mid T \in \widehat{\Gamma}\}$ is a family of balancing coefficients for the collection $\widehat{\Gamma}$. This implies that $\widehat{\Gamma}$ is a balanced Ω-collection.

Now from assertion (A) shown above applied to the balanced Ω-collection $\widehat{\Gamma}$ it follows that

$$\sum_{T \in \widehat{\Gamma}} \delta_T v_\Omega(T) \leqq v_\Omega(N).$$

We conclude from (A) and (B) that indeed the game v_Ω satisfies the Bondareva-Shapley conditions for both minimal balanced Ω-collections as well as balanced non-Ω-collections, implying that $C(v, \Omega) = C(v_\Omega) \neq \varnothing$. This proves the assertion.

Proof of Theorem 2.33

(i) *implies* (ii)
Let Ω be a lattice on N. Define $H\colon N \to 2^N$ by

$$H(i) = \{j \in N \mid i \in \partial_j \Omega\} \setminus \{i\} \tag{2.33}$$

Evidently H is a permission structure on N. We proceed by showing that $\Omega = \Omega_H$.

Recall that $\partial\Omega$ is the smallest basis of Ω. We show that $\partial\Omega$ is also a basis of Ω_H implying the desired assertion.

Let $i \in N$. Now for $j \notin \partial_i \Omega$ it holds that $\partial_i \Omega \cap H(j) = \varnothing$, since otherwise for $h \in \partial_i \Omega \cap H(j)$: $j \in \partial_h \Omega \subset \partial_i \Omega$.

Next let $S \in \Omega_H$. Clearly for every $j \in S$ and $i \in \partial_j \Omega$ with $i \neq j$ we have that $j \in H(i)$. Thus, $S \cap H(i) \neq \varnothing$, and, therefore, $i \in S$. As a consequence, $\partial_j \Omega \subset S$ for every $j \in S$. Therefore, $S = \cup_{j \in S} \partial_j \Omega$. This indeed shows that $\partial\Omega$ is a basis of Ω_H.

(ii) *implies* (i)
It is easy to see that $\varnothing \in \Omega_H$ as well as $N \in \Omega_H$.

Next let $S, T \in \Omega_H$. Regarding $S \cup T$ we remark that $N \setminus (S \cup T) \subset N \setminus S$. Hence,

$$(S \cup T) \cap H(N \setminus (S \cup T)) \subset T \cup \left[S \cap H(N \setminus S)\right] = T \cap \varnothing = \varnothing$$

showing that $S \cup T \in \Omega_H$.

Next consider $S \cap T$. Since $N \setminus (S \cap T) = (N \setminus S) \cup (N \setminus T)$ it easily follows that

$$H(N \setminus (S \cap T)) = H(N \setminus S) \cup H(N \setminus T).$$

This in turn implies that

$$\begin{aligned}(S \cap T) \cap \left[H(N \setminus (S \cap T))\right] &= (S \cap T) \cap \left[H(N \setminus S) \cup H(N \setminus T)\right] \\ &= \left[(S \cap T) \cap H(N \setminus S)\right] \cup \left[(S \cap T) \cap H(N \setminus T)\right] \\ &\subset \left[S \cap H(N \setminus S)\right] \cup \left[T \cap H(N \setminus T)\right] \\ &= \varnothing \cup \varnothing = \varnothing.\end{aligned}$$

This shows that $S \cap T \in \Omega_H$.

Proof of Theorem 2.34

Let Ω be a lattice on N. Obviously, for every $i \in N$ and $j \in \partial_i\Omega$ the payoff vector $e_i - e_J$ is in Γ_Ω. Furthermore Γ_Ω is a cone. This leaves us to prove that

$$\Gamma_\Omega \subset \text{ Cone } \left\{e_j - e_i \mid i \in N \text{ and} j \in \partial_i\Omega \right\}.$$

Let $x \in \Gamma_\Omega$ with $x \neq 0$. Select $j \in N$ to be such that $x_j > 0$.

Claim: There exists a player $i \in N$ with $x_i < 0$ and $j \in \partial_i\Omega$ such that there is some $\varepsilon > 0$ with

$$x' = x - \varepsilon(e_j - e_i) \in \Gamma_\Omega.$$

Proof Notice that from this it follows that $x'(S) = x(S) - \varepsilon$ for every $S \in \Omega$ with $i \notin S$ and $j \in S$.[10] Hence, whenever $x(S) = 0$ the situation $i \notin S$ and $j \in S$ is not allowed to occur. This implies that i has to be chosen in

$$T = \bigcap \left\{ S \in \Omega \;\middle|\; j \in S \text{ and} \sum_{i \in S} x_i = 0 \right\} \in \Omega.$$

Since $N \in \Omega$ and $x(N) = 0$, T is well defined and non-empty. Furthermore, $x(T) = 0$ since for any $V, W \in \Omega$ with $x(V) = x(W) = 0$ we have that $V \cap W,\ G \cup W \in \Omega$. Thus, $x(V \cap W) \geqq 0$ as well as $x(V \cup W) \geqq 0$. Together with

$$x(V \cap W) + x(V \cup W) = x(V) + x(W) = 0$$

this implies that $x(V \cap W) = x(V \cup W) = 0$.[11]

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[10] For convenience we define $x(S) = \sum_S x_i$ for any vector $x \in \mathbb{R}^N$ and coalition $S \in \Omega$.

[11] Actually we have shown that the collection $\{V \in \Omega \mid x(V) = 0\}$ is a lattice on N as well.

Since $j \in T$, $x_j > 0$ and $x(T) = 0$, there is a player $i \in T$ with $x_i < 0$. Suppose now that for any $i \in T$ with $x_i < 0$: $j \notin \partial_i\Omega$. Then

$$T' = \cup \{\partial_i\Omega \mid i \in T \text{ and} x_i < 0\} \in \Omega$$

and $T' \subset T$. Furthermore, $x(T') \geqq 0$ and, hence,

$$0 < x_j \leqq x(T \setminus T') = -x(T') \leqq 0,$$

which is a contradiction. Therefore we conclude that there exists a player $i \in T$ with $x_i < 0$ and $j \in \partial_i\Omega$.

Consider for i the payoff vector given by $x' = x - \varepsilon(e_j - e_i)$ with

$$\varepsilon = \min \left[\{x_j, -x_j\} \cup \{x(S) \mid S \in \Omega \text{ with } j \in S \text{ and} i \notin S\} \right]$$

By definition of T and the choice of $i \in T$ it is clear that $x(S) > 0$ for all $S \in \Omega$ with $j \in S$ and $i \notin S$. Therefore, $\varepsilon > 0$.

Next we prove that $x' \in \Gamma_\Omega$. First we observe that $x'(N) = 0$. Now for every $S \in \Omega$ we distinguish the following cases:

(i) $i \in S$ and $j \in S$: Then $x'(S) = x(S) \geqq 0$.
(ii) $i \in S$ and $j \notin S$: This case cannot occur since $j \in \partial_i\Omega$.
(iii) $i \notin S, j \in S$ and $x(S) = 0$: This case cannot occur either since $i \in T$.
(iv) $i \notin S, j \in S$ and $x(S) > 0$: Then $x'(S) = x(S) - \varepsilon \geqq 0$ by choice of ε.
(v) $i \notin S$ and $j \notin S$: Then $x'(S) = x(S) \geqq 0$.

We may also conclude for arbitrary $S \in \Omega$ that $x'(S) \leqq x(S)$ since the case that $i \in S$ and $j \notin S$ does not occur. Thus,

$$\#\{S \in \Omega \mid x'(S) = 0\} \geqq \#\{S \in \Omega \mid x(S) = 0\} \tag{2.34}$$

and

$$\#\{h \in N \mid x'_h = 0\} \geqq \#\{h \in N \mid x_h = 0\}. \tag{2.35}$$

The variable ε now can be chosen such that one of the inequalities (2.34) and (2.35) has to be strict.

This proves the claim.

We may proceed with the proof of the assertion by repeatedly applying the claim to generate a finite sequence of payoff vectors $x_0 = x, x_1, \ldots, x_K$ with for every $k \in \{0, \ldots, K-1\}$

$$x_{k+1} = x_k - \varepsilon_k \left(e_{j_k} - e_{i_k}\right) \text{ where } j_k \in \partial_{i_k}\Omega.$$

The sequence is terminated whenever $x_K = 0$. Now we conclude

$$x = \sum_{k=0}^{K-1} \varepsilon_k \left(e_{j_k} - e_{i_k} \right) \in \text{Cone} \left\{ e_j - e_i \mid i \in N \text{ and } j \in \partial_i \Omega \right\}$$

This completes the proof of the assertion stated in Theorem 2.34.

Proof of Theorem 2.36

(i) *implies* (ii)
Let $i, j \in N$ with $i \in S$ and $j \notin S$ for some $S \in \Omega$. Then $\partial_i \Omega \subset S$ and $\partial_j \Omega \not\subseteq S$. This implies that $\partial_i \Omega \neq \partial_j \Omega$.

(ii) *implies* (iii)
Suppose by contradiction that there exists $S \in \Omega$ with for every $i \in S$: $S \setminus \{i\} \notin \Omega$. Then $\cup_{j \in S:\, j \neq i} \partial_j \Omega \in \Omega$ contains $S \setminus \{i\}$ and is contained in S. So, $S = \cup_{j \in S:\, j \neq i} \partial_j \Omega$. Hence, for every $i \in S$ there exists some $j \in S \setminus \{i\}$ with $i \in \partial_j \Omega$.

Since S is finite there has to exist a sequence $i_1, \ldots, i_K$ in S with $i_K \in \partial_{i_1} \Omega$ and $i_{k+1} \in \partial_{i_k} \Omega$ for all $k \in \{1, \ldots, K-1\}$. By definition, therefore, $\partial_{i_1} \Omega = \partial_{i_2} \Omega = \cdots = \partial_{i_K} \Omega$. This is a contradiction to (ii).

(iii) *implies* (i)
This follows by definition by repeated application of (iii) to a coalition $S \in \Omega$ containing player i.

(ii) *is equivalent to* (iv)
By Theorem 2.33 there exists a permission structure H such that $\Omega = \Omega_H$. It is easy to see that H is acyclic if and only if $\partial_i \Omega_H \neq \partial_j \Omega_H$ for all $i \neq j$. This implies the equivalence.

(ii) *implies* (v)
First, $j \in \partial_i \Omega \setminus \{i\}$ implies that $\partial_j \Omega \subset \partial_i \Omega$. Now $i \notin \partial_j \Omega$ since otherwise $\partial_i \Omega \subset \partial_j \Omega$ implying equality in turn, and, hence, contradicting (ii). Therefore

$$\partial_i \Omega \setminus \{i\} = \bigcup_{j \in \partial_i \Omega:\ i \neq j} \partial_j \Omega \in \Omega.$$

(v) *implies* (vi)
We show that (v) implies that Γ_Ω is a pointed cone, thereby, through application of Proposition 2.23 showing that Ω is non-degenerate.

Suppose there exists $y \in \mathbb{R}^N$ with $\{y, -y\} \subset \Gamma_\Omega$. Then $y(N) = -y(N) = 0$ and $y(S) \geqq 0$ as well as $-y(S) \geqq 0$ for all $S \in \Omega$. Thus for any $i \in N$: $y(\partial_i \Omega) = 0$ and by (v) it follows that $y(\partial_i \Omega \setminus \{i\}) = 0$. Thus,

$$y_i = y(\partial_i \Omega) - y(\partial_i \Omega \setminus \{i\}) = 0$$

(vi) *implies* (ii)
Let $i, j \in N$. If $\partial_i \Omega = \partial_j \Omega$, then by Theorem 2.34 both $e_i - e_j$ and $e_j - e_i$ are in Γ_Ω. Hence, Γ_Ω is not a pointed cone. This would contradict that (vi) implies that Γ_Ω is a pointed cone.

Proof of Theorem 2.41

Before we develop the proofs of the two assertions stated in Theorem 2.41 we state and proof two intermediate insights. Recall first that Ω is a discerning lattice on N. Also, let H be an acyclic permission structure such that $\Omega = \Omega_H$.

Lemma 2.47 *Let $S \in \Omega$, $S \neq \varnothing$. Then $\Omega(S) = \Omega \cap 2^S$ and $\Omega' = \{N \setminus T \mid T \in \Omega\}$ are both discerning lattices on S and N, respectively.*

Proof Both $\Omega(S)$ and Ω' are lattices. Clearly $\Omega(S)$ contains a Weber string, implying that $\Omega(S)$ is discerning.

Also, for all $i, j \in N$ we may assume without loss of generality that there exists a coalition $S \in \Omega$ with $i \in S$ and $j \notin S$. Hence, $i \notin N \setminus S$ and $j \in N \setminus S$. Thus, Ω' is discerning. ■

Lemma 2.48 *For each coalition $S \in \Omega$ there is a Weber string in Ω of which S is one of its members.*

Proof Let $S \in \Omega$. By Lemma 2.47 there exist Weber strings in $\Omega(S)$ as well as $\Omega'(N \setminus S)$, say $(R'_i)_{i \in S}$ and $(R''_j)_{j \in N \setminus S}$, respectively. Then $(R_i)_{i \in N}$ is a Weber string in Ω where

$$R_i = \begin{cases} R'_i & \text{for } i \in S \\ \{i\} \cup (N \setminus R''_i) & \text{for } i \notin S \end{cases}$$

Clearly if $i \in S$, $R_i \in \Omega$; furthermore, $i \in R''_i = R_i$ and for i such that $R_i = R'_i \neq \{i\}$ there is a player $j \in S$ with $R - i \setminus \{i\} = R''_i \setminus \{i\} = R'_j = R_j$. Now if $i \notin S$ either $R_i = \{i\} \cup N \setminus R''_i = N \setminus (R''_i \setminus \{i\}) = N \setminus R''_j$ for a player $j \notin S$, or $R_i = N \setminus \varnothing = N$ and, since $(\Lambda')' = \Lambda$ for every lattice Λ, we conclude that $R_i \in \Omega$.

Furthermore, $i \in R_i$ and

- if $R''_i = N \setminus S$, then $R_i \setminus \{i\} = N \setminus R''_i = S = R'_j = R_j$ for some $j \in S$;
- if $R''_i \neq N \setminus S$ then there is a player $k \notin S$ with $R''_i = R''_k \setminus \{k\}$. Hence, $R_i \setminus \{i\} = N \setminus R''_i = N \setminus (R''_k \setminus \{k\}) = \{k\} \cup N \setminus R''_k = R_k$.

We conclude that for all players $i \in N$ we have $R_i \in \Omega$ and if $R_i \neq \{i\}$ there is a player j such that $R_i \setminus \{i\} = R_j$. ■

The following property now follows immediately from Lemmas 2.47 and 2.48:

Corollary 2.49 *Let $S_1, \ldots, S_K$ be elements in Ω such that $S_1 \subset S_2 \subset \cdots \subset S_K$. Then there exists a Weber string in Ω of which $S_1, \ldots, S_K$ are members.*

Proof of 2.41(a)
Since Ω is discerning by Lemma 2.39 there exists at least one Weber string in Ω.

Now suppose to the contrary that there exists a Core allocation $x \in C(\Omega, v)$ such that $x \notin \mathcal{W}(\Omega, v) + \Gamma_\Omega$. By Theorem 2.34 the set $\mathcal{W}(\Omega, v) + \Gamma_\Omega$ is polyhedral and since Ω is discerning/non-degenerate, it does not contain a nontrivial linear subspace. Thus x is an extreme point of

$$\mathrm{Conv}\;[\,\{x\} \cup (\mathcal{W}(\Omega, v) + \Gamma_\Omega)\,]\,.$$

The normals of all supporting hyperplanes in an extreme point of a polyhedral set are well known to form a full dimensional cone. Therefore, there exists a normal vector, say $p \in \mathbb{R}^N$, with non-equal coefficients such that for each $y \in \mathcal{W}(\Omega, v)$ and $y' \in \Gamma_\Omega$:

$$p \cdot x < p \cdot (y + y').$$

Now by Theorem 2.34 for every $i \in N, j \in \partial_i\Omega$ and $M \geqq 0$ it holds that for every $y \in \mathcal{W}(\Omega, v)$:

$$p \cdot x < p \cdot y + M p \cdot (e_j - e_i) = p \cdot y + M\,(p_j - p_i) \tag{2.36}$$

implying that for every $i \in N$ and $j \in \partial_i\Omega$: $p_j \geqq p_i$. Now label the players in N such that $p_1 > p_2 > \cdots > p_n$. Consider for every $k \in N$ the coalition $S_k = \{1, \ldots, k\}$. If $i \in S_k$, then $p_i \geqq p_k$, in turn implying that $p_j \geqq p_i \geqq p_k$ for all $j \in \partial_i\Omega$. Hence, $\partial_i\Omega \subset S_k$ for all $i \in S_k$ implying that

$$S_k = \bigcup_{i \in S_k} \partial_i\Omega \in \Omega.$$

Hence, $(S_k)_{k \in N}$ determines a Weber string in Ω. The corresponding marginal vector y for v now belongs to $\mathcal{W}(\Omega, v)$. Thus,

$$\begin{aligned} p \cdot x < p \cdot y &= \sum_{k \in N} \left[v(S_k) - v(S_{k-1}) \right] \\ &= v(N)\,p_n + \sum_{k \in N} v(S_k)\,(p_k - p_{k+1}). \end{aligned}$$

Since $x \in C(\Omega, v)$ and $S_k \in \Omega$ $(k \in N)$, we conclude that $x(N) = v(N)$ and $x(S_k) \geqq v(S_k)$, $k \neq n$. Furthermore, $p_k - p_{k+1} \geqq 0$ for $k \neq n$. Therefore, the above may be rewritten as

$$\begin{aligned}
p \cdot x &< v(N)\, p_n + \sum_{k \in N} v(S_k)\,(p_k - p_{k+1}) \\
&\leqq x(N)\, p_n + \sum_{k=1}^{n-1} \sum_{j=1}^{k} x_j\,(p_k - p_{k+1}) \\
&= \sum_{k=1}^{n} \sum_{j=1}^{k} x_j\, p_k - \sum_{k=2}^{n} \sum_{j=1}^{k-1} x_j\, p_k \\
&= x_1\, p_1 + \sum_{k=2}^{n} p_k\, x_k = p \cdot x.
\end{aligned}$$

This constitutes a contradiction, proving the desired inclusion.

Proof of 2.41(b)
Only if:
Suppose that $v \in \mathcal{G}^N$. Given the already established facts, we only have to show that every Weber allocation $x \in \mathcal{W}(\Omega, v)$ belongs to $C(\Omega, v)$. So, let $x \in \mathcal{W}(\Omega, v)$ correspond to some Weber string $(R_i)_{i \in N}$ with for every $i \in N$

$$x_i = v(R_i) - v(R_i \setminus \{i\}).$$

Since v is convex on Ω for every $i \in S \in \Omega$ we have that

$$v(R_i) + v(S \cap (R_i \setminus \{i\})) \geqq v(S \cap R_i) + v(R_i \setminus \{i\}).$$

Thus, for every $i \in S$:

$$x_i \geqq v(S \cap R_i) - v(S \cap R_i \setminus \{i\}).$$

Without loss of generality we may assume that N is labelled such that the players in S are labelled first and $\{1, \ldots, k\} \subset R_k$ for all $k \in S$. Then

$$\begin{aligned}
\sum_{i \in S} x_i \geqq \sum_{k=1}^{|S|} x_k &\geqq \sum_{k=1}^{|S|} v(S \cap R_k) - v(S \cap R_k \setminus \{k\}) \\
&= \sum_{k=1}^{|S|} v(\{1, \ldots, k\}) - \sum_{k=1}^{|S|-1} v(\{1, \ldots, k\}) \\
&= v(\{1, \ldots, |S|\}) = v(S).
\end{aligned}$$

Since S is chosen arbitrarily, we may conclude that $x \in C(\Omega, v)$.

If:

Suppose that $C(\Omega, v) = \mathcal{W}(\Omega, v) + \Gamma_\Omega$. Take $S, T \in \Omega$. Since Ω is discerning by Corollary 2.49 there exists a Weber string in Ω, say $(R_i)_{i \in N}$, such that it contains the coalitions $S \cup T$ and $S \cap T$.

Let $x \in \mathcal{W}(\Omega, v)$ be the corresponding Weber allocation—or marginal payoff vector. It is evident that $x(S \cap T) = v(S \cap T)$ as well as $x(S \cup T) = v(S \cup T)$. By hypothesis $x \in C(\Omega, v)$, which leads to the conclusion that

$$\begin{aligned} v(S) + v(T) &\leqq x(S) + x(T) = \\ &= x(S \cup T) + x(S \cap T) = \\ &= v(S \cup T) + v(S \cap T). \end{aligned}$$

This proves the convexity of v.

2.5 Problems

Problem 2.1 Let $N = \{1, 2, 3\}$ and consider the game $v \in \mathcal{G}^N$ given by the following table:

S	$\varnothing$	1	2	3	12	13	23	123
$v(S)$	0	0	5	6	15	0	10	20

(a) Give the $(0, 1)$-normalization v' of this game v.
(b) Compute the Core $C(v) \subset I(v)$ of the game v and of its $(0, 1)$-normalization v'. Show that these two Cores are essentially identical. Formulate this identity precisely.
(c) Draw the Core $C(v')$ of the $(0, 1)$-normalization v' of the game v in the two-dimensional simplex $\mathcal{S}^2$.

Problem 2.2 Consider the so-called *bridge game*, introduced by Kaneko and Wooders (1982, Example 2.3). Let $N = \{1, \dots, n\}$ and define $v \in \mathcal{G}^N$ by

$$v(S) = \begin{cases} 1 & \text{if } |S| = 4 \\ 0 & \text{otherwise.} \end{cases}$$

In the bridge game only quartets can generate benefits.

Show that $C(v) \neq \varnothing$ if and only if $n = 4m$ for some $m \in \mathbb{N}$.

Problem 2.3 Construct a *non-essential* three player game that has an *empty* Core. Check the emptiness by depicting the three inequalities (2.5)–(2.7) for the $(0, 1)$-normalization of this game in the two-dimensional set of imputations of this game.

Problem 2.4 Construct an essential three player game that has a Core consisting of exactly *one* imputation. Check again the required uniqueness property by depicting the three inequalities (2.5)–(2.7) for the $(0, 1)$-normalization of this game in the two-dimensional simplex of imputations $\mathcal{S}^2$.

Problem 2.5 Construct a cooperative game that is essential—but not superadditive—such that its Core does not coincide with the set of undominated imputations, i.e., such that the Core is a strict subset of the set of undominated imputations.

Problem 2.6 Construct a detailed proof of Corollary 2.11 based on the original Bondareva-Shapley Theorem rather than Corollary 2.13.

Problem 2.7 Construct a detailed proof of Corollary 2.13.

Problem 2.8 Let v be an essential three player game $v \in \mathcal{G}^N$ with $N = \{1, 2, 3\}$. Show that $C(v)$ is a singleton—i.e., $\#C(v) = 1$—if and only if $v(12)+v(13)+v(23) = 2v(N)$.

Problem 2.9 One can also consider the case of "costly" blocking by a coalition. Let $v \in \mathcal{G}^N$. We introduce $\varepsilon \in \mathbb{R}$ as the cost for a coalition $S \subset N$ to block any imputation $x \in I(v)$.[12] This leads to the introduction of the *ε-Core* $C_\varepsilon(v) \subset I(v)$ defined by $x \in C_\varepsilon(v)$ if and only if

$$\sum_{i \in S} x_i \geqq v(S) - \varepsilon$$

for every non-empty coalition $S \subset N$.

(a) Show that for every game $v \in \mathcal{G}^N$ if $\varepsilon_1 < \varepsilon_2$, then $C_{\varepsilon_1}(v) \subset C_{\varepsilon_2}(v)$.
(b) Show that there exists a minimal cost parameter $\varepsilon \in \mathbb{R}$ such that $C_\varepsilon(v) \neq \varnothing$. This particular $C_\varepsilon(v)$ is known as the *least Core* of v indicated by $C_L(v)$.
(c) Show that the interior of $C_L(v)$ is empty. In particular draw the least Core of the $(0, 1)$-normalization of the trade game discussed in Examples 1.9 and 1.19.
(d) let $v, w \in \mathcal{G}^N$. Suppose that for ε_v and ε_w it holds that

$$C_{\varepsilon_v}(v) = C_{\varepsilon_w}(w) \neq \varnothing.$$

Show that for any $\delta > 0$ it now has to hold that

$$C_{\varepsilon_v - \delta}(v) = C_{\varepsilon_w - \delta}(w).$$

In particular show that $C_L(v) = C_L(w)$.

[12] Remark that $\varepsilon < 0$ is possible. This implies that the cost of blocking is negative, hence there are synergy effects or other benefits to the act of blocking.

Problem 2.10 Consider the communication network on $N = \{1, 2, 3, 4, 5, 6, 7\}$ depicted in Fig. 2.6 on page 42. Let $\Omega \subset 2^N$ be the collection of formable coalitions in this network in the sense of Myerson (1977). Compute the Ω-Core of the unanimity game u_{17}.

Problem 2.11 Give a proof of Proposition 2.21.

Problem 2.12 Consider the proof of Proposition 2.23. There it is stated that "Γ_Ω is a pointed cone if and only if for any $x \in \mathbb{R}^N$ with $\sum_S x_i = 0$ for all $S \in \Omega$ we must have $x = 0$." Show this property in detail.

Problem 2.13 Construct a proof of Lemma 2.39(a) by using the properties stated in Theorem 2.36.

Problem 2.14 A vector $x \in \mathbb{R}^N$ is a *Dual-Core allocation* of the cooperative game $v \in \mathcal{G}^N$ if x satisfies the efficiency requirement

$$\sum_{i \in N} x_i = v(N) \tag{2.37}$$

and for every coalition $S \subset N$

$$\sum_{i \in S} x_i \leqq v(S). \tag{2.38}$$

Notation: $x \in C^\star(v)$.[13]

(a) We define the *dual game* of v by $v^\star \in \mathcal{G}^N$ with $v^\star(S) = v(N) - v(N \setminus S)$. Show that $C^\star(v) = C(v^\star)$.
(b) Define the Dual-Weber set in the same fashion as the Dual-Core. Be precise in your formulation. What is the relationship between the Dual-Weber set of a game and the Weber set of its dual game?

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[13] The Dual-Core is also called the "Anti-Core" by some authors. Since the definition is really based on a duality argument, I think it is more appropriate to refer to $C^\star$ as the Dual-Core.

Chapter 3
Axiomatic Value Theory

So far we have focussed our discussion on the Core and related set theoretic solution concepts as concepts that address the fundamental problem of cooperative game theory—the identification of stable binding agreements between the participating players. As such these set theoretic solution concepts consist of allocations that satisfy some fundamental properties of negotiating power of coalitions and, consequently, are founded on a description of the bargaining power of the coalitions of players.

In set theoretic solution concepts such as the Core and the Weber set a collection of allocations is identified that is in many ways "acceptable" from certain points of view. In this regard these allocations represent a certain *standard of behavior*.[1]

The development of the Core and related concepts as standards of behavior can be viewed as an attempt to simplify or reduce the fundamental problem of cooperative game theory. The next step in this reductionist process is to introduce multiple standards of behavior that a solution has to satisfy. This was seminally pursued by Shapley (1953) in his ground breaking contribution, establishing the field of *value theory*. He introduced three axioms, each essentially describing a simple behavioral rule or property. Subsequently he showed that the collection of allocations satisfying these three axioms forms a singleton for *every* cooperative game. Thus, he arrived at the notion of an "axiomatic value"; a single-valued solution concept that is guaranteed to exist for every game and which is completely characterized by a given set of behavioral axioms.

Since then cooperative game theory has developed very rapidly as an axiomatic theory of single valued solution concepts for situations represented by games in characteristic function form. The Shapley value in particular has been the subject of

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[1] The notion of a standard of behavior was already introduced by von Neumann and Morgenstern (1953) and resulted into the notion of the *vNM solution*. The vNM solution is a set of allocations that are stable with regard to the standard of behavior introduced by von Neumann and Morgenstern (1953). A standard of behavior can be used as the foundation of an alternative approach to interactive decisions in general. Greenberg (1990) pursued this further and used the foundations of the vNM solution as the basis of his theory. I will not further go into this matter and the vNM solution concept. I refer to Greenberg (1990) and Owen (1995, Chapter XI) for an elaborate discussion.

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extensive debate and a wide variety of axiomatizations of the Shapley value have been developed during the past fifty years.

The fundamental notion of a value is introduced through the next definition as a function that assigns to every game in characteristic function form exactly one allocation.

Definition 3.1 A *value* is a function $\phi\colon \mathcal{G}^N \to \mathbb{R}^N$ that assigns to every cooperative game v a single allocation $\phi(v) \in \mathbb{R}^N$.

In this chapter we consider in particular so-called axiomatic values. A value is axiomatic if there exist a finite set of well-defined properties or *axioms* that fully characterize it. This means that the given axioms are independent in the sense that they do not imply each other and that the set of chosen axioms exactly generates the formulated value for every cooperative game. Such sets of characterizing axioms for a particular value are also known as *axiomatizations* of that value.

Subsequently the literature on cooperative game theory developed more axiomatic values by selecting different sets of axioms. Next game theorists developed axiomatic values on certain specific classes of cooperative games $\mathcal{H} \subset \mathcal{G}^N$. If restrictions on the domain of a value are applied, one usually indicates these solutions as "indices" rather than values. Here I mention two examples of indices on the class of simple cooperative games: The Shapley-Shubik index (Shapley and Shubik, 1954) and the Banzhaf power index (Banzhaf, 1965, 1968).

3.1 Equivalent Formulations of the Shapley Value

The seminal axiomatic value introduced by Shapley (1953) has had a very special appeal. First, Shapley characterized this value through three very straightforward and appealing axioms. Second, the value's computation is rather straightforward and can be done in multiple ways, including some rather appealing computational methods. Third, this value has a wide ranging applicability as shown by subsequent work. These three reasons make the Shapley value a lasting contribution to the theory of cooperation in interactive decision situations.

There have been developed numerous formulations of the Shapley value. In this short section I introduce the main formulations and present them independently from the axiomatizations that have been proposed by different authors.

I first discuss computationally the most convenient formulation to describe the Shapley value.

Definition 3.2 The *Shapley value* is the value function $\varphi\colon \mathcal{G}^N \to \mathbb{R}^N$, which for every player $i \in N$ is given by

$$\varphi_i(v) = \sum_{S \subset N\colon i \in S} \frac{\Delta_v(S)}{|S|} \tag{3.1}$$

If one considers a Harsanyi dividend to be the true productive value that a coalition generates, then the Shapley value is just the fair division of all these dividends among the members of the value-generating coalitions.

From this formulation it is already clear that the Shapley value combines power, feasibility and fairness. Indeed, a coalition is assigned its dividend, which represents its power as well as the collective value that is feasible for a coalition to generate. The equal division of the dividend over the members of the value-generating coalition refers to the fairness component in the Shapley value.

To illustrate the computational ease of this formulation we look at a simple example.

Example 3.3 Consider $N = \{1,2,3\}$. Consider that these three players are countries in a negotiation procedure. Country 1 has 5 votes, Country 2 has 4 votes, and Country 3 has 1 vote. To "win" a coalition of countries needs 6 votes. We give winning a value of 100, but consensus gives an additional payoff of 40. How should the honor points of a total $100 + 40 = 140$ that are attained in consensus be divided?

Formally we let v on N be given in the following table. This table also includes the Harsanyi dividends for every coalition in this game.

S	$\varnothing$	1	2	3	12	13	23	123
$v(S)$	0	0	0	0	100	100	0	140
$\Delta_v(S)$	0	0	0	0	100	100	0	−60

The Shapley value for this game is now given by

$$\varphi_1(v) = \frac{100}{2} + \frac{100}{2} + \frac{-60}{3} = 80$$
$$\varphi_2(v) = \frac{100}{2} + \frac{-60}{3} = 30$$
$$\varphi_2(v) = \frac{100}{2} + \frac{-60}{3} = 30$$

It is clear that Country 1 is the most powerful country in this negotiation game. However, surprisingly Country 2 is apparently as weak as Country 3 according to their respective Shapley values. This is due to the problem that Country 2 does not have enough votes to block Country 1 by cooperating with Country 3.

As a remark, I mention that the Core of this voting situation is given by a convex polyhedron with four corner points

$$C(v) = \text{Conv}\,\{(140,0,0),\ (100,40,0),\ (100,0,40),\ (60,40,40)\}.$$

In the Core, the powerful position of Country 1 is clearly depicted. In this case the Shapley value is in the Core, although it is not equal to its center given by (100,20,20). ■

Next consider the trade example that I developed in Examples 1.2, 1.5, and 1.9.

Example 3.4 I refer to Example 1.9 for the following description of the three person utilitarian trade game v_u:

S	$\varnothing$	1	2	3	12	13	23	123
$v_u(S)$	0	0	0	0	100	150	0	150
$\Delta_{v_u}(S)$	0	0	0	0	100	150	0	-100

The Shapley value for this game v_u is now given by

$$\varphi_1(v_u) = \frac{100}{2} + \frac{150}{2} + \frac{-100}{3} = 91\tfrac{2}{3}$$
$$\varphi_2(v_u) = \frac{100}{2} + \frac{-100}{3} = 16\tfrac{2}{3}$$
$$\varphi_2(v_u) = \frac{150}{2} + \frac{-100}{3} = 41\tfrac{2}{3}$$

In this trade example it is interesting to see that player 2 is assigned a positive payoff in the Shapley value. This is due to the fact that player 1 can trade with player 2 and generate a positive surplus. This gives player 2 a fair claim on the dividend of the trade coalition 12. This results in the Shapley value computed.

As a comparison, the Core of v_u can be determined as

$$C(v_u) = \left\{ (x_1, x_2, x_3) \;\middle|\; 100 \leqq x_1 \leqq 150,\ x_2 = 0,\ x_3 = 150 - x_1 \right\}$$

Certainly $\varphi(v_u) \notin C(v_u)$. The Core essentially reflects that player 2 is completely powerless, while as pointed out above, player 2 is still recognized in the Shapley value. ■

The relationship between the Shapley value and the Core of a cooperative game is a rather interesting point of discussion. Above I discussed examples in which the Shapley value is in the Core (Example 3.3) and the Shapley value is outside a non-empty Core (Example 3.4). The next example discusses a game with an empty Core. In that case, the Shapley value is well-defined, but is situated separately from the Core.

Example 3.5 Again consider $N = \{1, 2, 3\}$. The generated values of the various coalitions in this three-player game are depicted in the next table:

S	$\varnothing$	1	2	3	12	13	23	123
$v(S)$	0	6	0	0	10	10	10	15
$\Delta_v(S)$	0	6	0	0	4	4	10	-9

From the given coalitional values, it is clear that the Core is indeed empty, i.e., $C(v) = \varnothing$. This is due to the singular, individual claim of 6 units that player 1 can hold out for in conjunction with the claim of coalition 23 of 10 units; both cannot be satisfied from the total generated 15 units. On the other hand, the Shapley value of this particular game is given by $\varphi(v) = (7, 4, 4)$.

The difference between the Core and the Shapley value here is that the Core only considers the negotiation power of the various coalitions. In this game, the 2-player coalitions 12, 13, and 23 have conflicting powers, resulting in the emptiness of the Core. The Shapley value divides the generated Harsanyi dividends regardless of the bargaining power of these coalitions, resulting into a plausible imputation. ■

The formulation of the Shapley value that I give in (3.1) is directly related to the selectors and the allocations based on these selectors, underlying the Selectope. The following result is a straightforward consequence of the definitions of selector allocations and the Shapley value.

Corollary 3.6 *The Shapley value $\varphi(v)$ of $v \in \mathcal{G}^N$ is the average of all selector allocations of v, i.e.,*

$$\varphi_i(v) = \frac{1}{|A_N|} \sum_{\alpha \in A_N} m_i^{\alpha}(v), \tag{3.2}$$

where $A_N = \{\alpha \mid \alpha\colon 2^N \to N\}$ is the family of all selectors on N.

The following theorem provides an overview of the other main formulations of the Shapley value. These formulations are widely adopted and used in applications of this theory.

Theorem 3.7 *Let $v \in \mathcal{G}^N$ and $i \in N$. Then the following statements are equivalent:*

(i) $\varphi_i(v) \in \mathbb{R}$ *is the Shapley value of player i in game v.*

(ii) **The standard formulation**

$$\varphi_i(v) = \sum_{S \subset N\colon i \in S} \frac{(|S|-1)!\,(n-|S|)!}{n!} \left[\, v(S) - v(S \setminus \{i\}) \,\right] \tag{3.3}$$

(iii) **The probabilistic formulation**

$$\varphi_i(v) = \frac{1}{n!} \sum_{\rho \in \Upsilon(2^N)} x^{\rho} \tag{3.4}$$

where $\Upsilon(2^N)$ is the collection of Weber strings in 2^N, $n! = |\Upsilon(2^N)|$ is the number of Weber strings in 2^N, and x^{ρ} is the marginal allocation corresponding to the Weber string $\rho \in \Upsilon(2^N)$.

(iv) **The MLE formulation**

$$\varphi_i(v) = \int \frac{\partial E_v}{\partial x_i}(t, \ldots, t)\, dt \tag{3.5}$$

For a proof of Theorem 3.7 I refer to the appendix of this chapter.

3.2 Three Axiomatizations of the Shapley Value

One of the main goals of value theory is to identify a set of axioms that uniquely determines the value under consideration. Furthermore, it is the ultimate goal of this theory to identify axioms that are independent of each other. Independence can be shown by identifying other values that satisfy all axioms except one under consideration. Independence can be interpreted as the property that the collection of axioms is "minimal"; there are no unnecessary properties within the class of stated axioms. In the case of independent axioms one says that this class of axioms characterizes the value under consideration. In that case the axioms form an *axiomatization* of that particular value.

I discuss three fundamentally different axiomatizations of the Shapley value φ on $\mathcal{G}^N$. The first axiomatization is the original axiomatization developed by Shapley (1953). It considers four axioms: Efficiency, the null-player property, symmetry[2] and additivity. These four axioms are widely recognized as the most fundamental ones in cooperative game theory. These four properties provide a proper axiomatization of the Shapley value.

Shapley's axiomatization instigated an extensive debate on the additivity property. This was considered the weakest point in Shapley's axiomatization. From its inception it has been considered a very strong property and revised axiomatizations were called for. The main axiomatization that does not use additivity is the second one that we consider here, developed by Young (1985). He considered a monotonicity property in which higher individual marginal contributions imply a higher payoff to the player in question. Young showed that his *strong monotonicity* property replaces both the dummy and additivity properties. Therefore, it avoids the use of the additivity property.

Our final axiomatization is a relatively recent contribution to the literature on cooperative game theory, filling a void in our understanding of the Shapley value, which is nevertheless very intuitive. Indeed, van den Brink (2001) bases his axiomatization on a very natural notion of fairness in the allocation of the generated wealth. His fairness axiom replaces additivity as well as symmetry. This axiomatization therefore gives an explicit formulation to the fairness that is present in the Shapley value as an allocation rule.

Before proceeding into the detailed discussion of these axiomatizations I introduce some auxiliary notions. These notions help us formulate the various properties.

- A player $i \in N$ is a *null-player* in the game $v \in \mathcal{G}^N$ if for every coalition $S \subset N$: $v(S) = v(S \setminus \{i\})$. Hence, a null-player is a non-contributor in a cooperative game.

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[2] Many sources in cooperative game theory also indicate the symmetry axiom as *anonymity*. Symmetry refers to an equal treatment property, while the term anonymity refers to the notion that the name of a player is not important; just what values each player generates determine the payoff to that player.

- Let $\rho\colon N \rightleftarrows N$ be any permutation on N. Then for every game $v \in \mathcal{G}^N$ by the *permutated game* $\rho v \in \mathcal{G}^N$ we mean the game such that for every coalition $S = \{i_1, \dots, i_K\}$

$$\rho v(\rho S) = v(S), \tag{3.6}$$

where $\rho S = \{\rho(i_1), \dots, \rho(i_K)\}$.
- Two players $i, j \in N$ are *equipoised* in the game $v \in \mathcal{G}^N$ if for every $S \subset N \setminus \{i, j\}$:

$$v(S \cup \{i\}) = v(S \cup \{j\}). \tag{3.7}$$

Equipoised players can replace each other without affecting the productive value generated by any coalition in question. Thus, for all cooperatively game theoretic purposes these players are completely equal.

3.2.1 Shapley's Axiomatization

Although Shapley's original axiomatization was formulated for a "universe" of players that contained any finite player set, we can easily reformulate his axiomatization for a given player set N. We first introduce the four axioms at the foundation of the axiomatization developed by Shapley (1953):

Efficiency: A value $\phi\colon \mathcal{G}^N \to \mathbb{R}^N$ is efficient if for every game $v \in \mathcal{G}^N$:

$$\sum_{i \in N} \phi_i(v) = v(N) \tag{3.8}$$

Null-player Property: A value $\phi\colon \mathcal{G}^N \to \mathbb{R}^N$ satisfies the null-player property if for every game $v \in \mathcal{G}^N$ it holds that $\phi_i(v) = 0$ for every null-player $i \in N$ in the game v.

Symmetry: A value $\phi\colon \mathcal{G}^N \to \mathbb{R}^N$ is symmetric if for every permutation $\rho\colon N \rightleftarrows N$:

$$\phi_{\rho(i)}(\rho v) = \phi_i(v) \tag{3.9}$$

Additivity: A value $\phi\colon \mathcal{G}^N \to \mathbb{R}^N$ is additive if for all games $v, w \in \mathcal{G}^N$ and every player $i \in N$:

$$\phi_i(v + w) = \phi_i(v) + \phi_i(w) \tag{3.10}$$

I first investigate the independence of these four fundamental properties. We use several examples of values to do this.

Example 3.8 For each of the four properties introduced above I discuss an example that satisfies the three other properties, but *not* the property in question.

Efficiency: Consider the value ψ^1 defined by

$$\psi_i^1(v) = \sum_{S \subset N:\, i \in S} \Delta_v(S) \tag{3.11}$$

This value is certainly not efficient for every game. We just state here that ψ^1 satisfies the null-player property, symmetry, as well as additivity.

The null-player property: Consider the *egalitarian value* ψ^2 given by

$$\psi_i^2(v) = \frac{v(N)}{n} \tag{3.12}$$

The egalitarian value satisfies efficiency, symmetry and additivity. However, it does not satisfy the null-player property.

Symmetry: Consider a selector $\alpha\colon 2^N \to N$ with $\alpha(S) \in S$ for all $S \neq \varnothing$. Now we define the value ψ^3 as the selector allocation m^α corresponding to α defined by

$$\psi_i^3(v) = m_i^\alpha(v) = \sum_{S:\, i=\alpha(S)} \Delta_v(S), \tag{3.13}$$

This value satisfies efficiency, the null-player property as well as additivity. With regard to the null-player property I remark that any player $i \in S$ is *not* a null-player if $\Delta_v(S) \neq 0$. Hence, $\psi_i^3(v) \neq 0$ only if i is a selector in a coalition S with $\Delta_v(S) \neq 0$, in turn implying that i is not a null-player in v.

Furthermore, the value ψ^3 does not satisfy the symmetry property. This is due to the effect of the selector α in its definition.

Additivity: A value that satisfies efficiency, the null-player property as well as symmetry, but does not satisfy the additivity property, is the *Nucleolus*. This concept was introduced by Schmeidler (1969) and is based on a model of a bargaining process. The bargaining conditions identify a unique allocation.

The nucleolus of a cooperative game is a rather complex concept. Consequently I will not discuss this concept in detail. Instead I limit myself to pointing out its existence and the fact that it is a value that shows that additivity is an independent requirement from the other three Shapley axioms in the standard axiomatization of the Shapley value.

The discussion provided in this example shows that these four listed properties are indeed independent from each other. ■

The main characterization of the Shapley value is provided in the following theorem. It states essentially that the four properties formulated in this section provide a proper axiomatization of the Shapley value. Since the proof of this fundamental insight is essential, we provide a constructive proof after the statement of the assertion.

Theorem 3.9 (Shapley, 1953) *The Shapley value φ is the unique value on $\mathcal{G}^N$ that satisfies efficiency, the null-player property, symmetry and additivity.*

Proof It is easy to check that the Shapley value φ satisfies the four properties that are listed in the assertion. I leave it to the reader to go through the analysis for this.

To show the reverse, recall first that, using the unanimity basis $\mathcal{U}$ of $\mathcal{G}^N$, every game $v \in \mathcal{G}^N$ can be written as

$$v = \sum_{S \subset N} \Delta_v(S)\, u_S. \tag{3.14}$$

Now let $\phi \colon \mathcal{G}^N \to \mathbb{R}^N$ be a value satisfying the four properties listed.

Let $S \subset N$ and $C \in \mathbb{R}$. Consider the game $w = C\, u_S$. Then all players $j \notin S$ are null-players in w, implying that $\phi_j(w) = 0$ for all $j \notin S$. On the other hand all players in S are equipoised and by symmetry this implies that $\phi_i(w) = \phi_j(w) = \bar{c}$ for all $i, j \in S$. Finally, by efficiency this now implies that

$$\sum_{i \in N} \phi_i(w) = \sum_{j \notin S} \phi_j(w) + \sum_{i \in S} \phi_i(w) = |S| \cdot \bar{c} \equiv w(N) = C. \tag{3.15}$$

Hence, we have derived that

$$\phi_i(w) = \phi_i\left(C\, u_S\right) = \begin{cases} \frac{C}{|S|} & \text{for } i \in S \\ 0 & \text{for } i \notin S \end{cases} \tag{3.16}$$

Thus we derive that for an arbitrary game v and arbitrary player $i \in N$ by the additivity property that

$$\begin{aligned} \phi_i(v) &= \sum_{S \subset N} \phi_i\left(\Delta_v(S)\, u_S\right) \\ &= \sum_{S \subset N\colon i \in S} \phi_i\left(\Delta_v(S)\, u_S\right) \\ &= \sum_{S \subset N\colon i \in S} \frac{\Delta_v(S)}{|S|} = \varphi_i(v) \end{aligned}$$

This indeed shows that ϕ necessarily is the Shapley value φ. ∎

The axiomatization developed in this section can be called the "standard" axiomatization of the Shapley value. Although the original axiomatization of Shapley (1953) was developed in a slightly different fashion, the four axioms of efficiency, the null-player property, symmetry and additivity are recognized as the foundational axioms that have to be attributed to Shapley.

An alternative development of a closely related axiomatization was pursued by Weber (1988). He introduced a step-wise development of the axioms and the resulting values. In this approach the Shapley value is developed as a very special case of a *probabilistic value*. Weber also relates in that paper the class of probabilistic values with random-order values, based on the Weber strings discussed in the previous chapter. I do not pursue this approach here, but refer to Weber (1988) instead.

3.2.2 Young's Axiomatization

Next I turn to the question whether the Shapley value can be axiomatized using a monotonicity property. There are several formulations of monotonicity possible. I explore the two most important ones here.

First we introduce Shubik's coalitional monotonicity condition (Shubik, 1962). This is the most intuitive and straightforward expression of the idea that higher generated wealth leads to higher payoffs for the players.

Definition 3.10 A value ϕ satisfies *coalitional monotonicity* if for all games $v, w \in \mathcal{G}^N$ with $v(T) \geqq w(T)$ for some coalition $T \subset N$ and $v(S) = w(S)$ for all $S \neq T$, then it holds that

$$\phi_i(v) \geqq \phi_i(w) \quad \text{for all } i \in T \tag{3.17}$$

It should be clear that the Shapley value indeed satisfies this coalitional monotonicity property. (For a discussion on this, I refer to the problem section of this chapter.) Unfortunately, the coalitional monotonicity property does not characterize the Shapley value. Worse, no "Core" value satisfies this property as was shown by Young (1985).

Proposition 3.11 (Young, 1985, Theorem 1) *If $n \geqq 5$, then there exists no value ϕ such that $\phi(v) \in C(v)$ for every game $v \in \mathcal{G}^N$ with $C(v) \neq \varnothing$ and that satisfies coalitional monotonicity.*

Proof Let $n = 5$, i.e., $N = \{1, 2, 3, 4, 5\}$. We construct now two games $v, w \in \mathcal{G}^N$ that show the desired property. Define the following coalitions:

$$S_1 = \{3, 5\} \quad S_2 = \{1, 2, 3\} \; S_3 = \{1, 2, 4, 5\}$$
$$S_4 = \{1, 3, 4\} \; S_5 = \{2, 4, 5\}$$

Now construct $v \in \mathcal{G}^N$ such that

$$v(S_1) = 3 \quad v(S_2) = 3 \quad v(S_3) = 9$$
$$v(S_4) = 9 \quad v(S_5) = 9 \quad v(N) = 11$$

and for all other coalitions S we define

$$v(S) = \begin{cases} \max_{S_k \subset S} v(S_k) & \text{if } S_K \subset S \text{ for some } k \in \{1,2,3,4,5\} \\ 0 & \text{otherwise} \end{cases}$$

Now $x \in C(v)$ if and only if

$$\sum_{i \in S_k} x_i \geqq v(S_k) \text{ for } k \in \{1,2,3,4,5\}.$$

Adding these inequalities results into $\sum_k \sum_{S_k} x_i = 3 \sum_N x_i \geqq 33$, implying that $\sum_N x_i \geqq 11$. But $\sum_N x_i = v(N) = 11$ by definition, which leads to the conclusions that all inequalities $\sum_{S_k} x_i \geqq v(S_k)$ must be equalities. These have a unique solution, $\bar{x} = (0,1,2,7,1)$. Now, $C(v) = \{\bar{x}\}$.

Now compare the game $w \in \mathcal{G}^N$ which is identical to v except that $w(S_3) = w(N) = 12$. A similar argument shows that there is a unique Core imputation given by $\hat{x} = (3,0,0,6,3)$. Hence, the allocation to players 2 and 4 has decreased, even though the value of the coalitions containing these two players has monotonically increased.

Now any Core value ϕ has to select the unique Core imputation for v and w. But then this value ϕ does not satisfy coalitional monotonicity by definition.[3] This easily extends to cases with $n \geqq 5$. ■

The problem with coalitional monotonicity is that it considers *absolute* changes in the wealth generated by the various coalitions. Instead we have to turn to a consideration of the *relative* changes in the wealth generated by coalitions. This is formulated in the notion of strong monotonicity.

Definition 3.12 A value ϕ satisfies *strong monotonicity* if for all games $v, w \in \mathcal{G}^N$ it holds that

$$D_i v(S) \geqq D_i w(S) \text{ for all } S \subset N \text{ implies } \phi_i(v) \geqq \phi_i(w) \tag{3.18}$$

where

$$D_i v(S) = \begin{cases} v(S) - v(S \setminus \{i\}) & \text{if } i \in S \\ v(S \cup \{i\}) - v(S) & \text{if } i \notin S \end{cases}$$

denotes the marginal contribution of a player to an arbitrary coalition $S \subset N$.

The following result is the main insight with regard to monotonicity and the Shapley value. It constitutes a very powerful axiomatization of the Shapley value.

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[3] I remark here that any monotonic modification of a game can be decomposed into a number of one-step modifications in which the value of a single coalition is monotonically increased. Coalitional monotonicity can be applied to each of these one-step modifications to arrive the conclusion that under monotonic modification the values of all players should monotonically increase as well.

Theorem 3.13 (Young, 1985, Theorem 2) *The Shapley value φ is the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry and strong monotonicity.*

For a proof of this theorem I refer to the appendix of this chapter.

3.2.3 van den Brink's Axiomatization

In this subsection I consider so-called "fairness" considerations with regard to the Shapley value. Explicit fairness axioms were first made in the context of allocating the benefits of a cooperative game among players that interact through a communication network. Myerson (1977) seminally introduced a fairness property to characterize his Myerson value for cooperative games with an exogenously given communication network. This was further developed by Myerson (1980) for non-transferable utility games with arbitrary cooperation structures.[4] Myerson imposed that the removal of a link affects both constituting players of that link in equal fashion. Hence, the marginal payoff to the deletion of a link is equal for the two players making up that particular link.

In his seminal contribution, van den Brink (2001) applied the same reasoning to any pair of equipoised players in an arbitrary game. Formally we recall the definition of two equipoised players. van den Brink's fairness property now requires that equipoised players are treated equal.

Definition 3.14 Let ϕ be some value on $\mathcal{G}^N$.

(a) The value ϕ satisfies the *equal treatment property* if for all players $i,j \in N$ and all games $v \in \mathcal{G}^N$:

$$i \text{ and } j \text{ are equipoised in } v \text{ implies } \phi_i(v) = \phi_j(v).$$

(b) The value ϕ is *fair* if for all players $i,j \in N$ and all games $v,w \in \mathcal{G}^N$:

$$i \text{ and } j \text{ are equipoised in } w \text{ implies } \phi_i(v+w) - \phi_i(v) = \phi_j(v+w) - \phi_j(v).$$

Given that equipoised players make equal contributions to coalitions, the equal treatment property requires the allocation of exactly the same payoff to either of these players. The fairness property on the other hand requires that, if a game in which two players are equipoised, is added to any other game, then these two players receive exactly the same marginal payoff from that addition.

The next proposition summarizes the relationships between these two fairness properties and the well-established Shapley axioms.

Proposition 3.15 *Let ϕ be some value on $\mathcal{G}^N$.*

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[4] For details I refer to the discussion of the Myerson value for arbitrary cooperation structures in Section 3.4 of this chapter.

(a) *ϕ satisfies the equal treatment property if and only if ϕ is symmetric.*
(b) *If ϕ satisfies symmetry and additivity, then ϕ also satisfies fairness.*
(c) *If ϕ satisfies the null-player property and fairness, then ϕ also satisfies symmetry.*

Proof

(a) This is rather straightforward and left as an exercise to the reader.
(b) Let ϕ satisfies symmetry and additivity. If $i, j \in N$ are equipoised in w, then for every v it holds that

$$\begin{aligned}\phi_i(v+w) - \phi_i(v) &= \phi_i(v) + \phi_i(w) - \phi_i(v) = \\ &= \phi_i(w) = \phi_j(w) = \\ &= \phi_j(v) + \phi_j(w) - \phi_j(v) = \phi_j(v+w) - \phi_j(v)\end{aligned}$$

thus ϕ indeed satisfies fairness.
(c) Let ϕ satisfies the null-player property and fairness. For the null game η the null-player property now implies that $\phi_i(\eta) = 0$ for all $i \in N$. If i and j are equipoised, then by fairness and the above we get

$$\phi_i(v) = \phi_i(v+\eta) - \phi_i(\eta) = \phi_j(v+\eta) - \phi_j(\eta) = \phi_j(v).$$

This indeed implies that ϕ is symmetric. ■

The fairness axiom can be implemented into a proper axiomatization of the Shapley value. The proof of the next theorem is relegated to the appendix of this chapter.

Theorem 3.16 (van den Brink, 2001, Theorem 2.5) *The Shapley value φ is the unique value on $\mathcal{G}^N$ that satisfies efficiency, the null-player property and fairness.*

Without discussion I also mention here that the fairness hypothesis is independent of the two other axioms in this axiomatization.[5]

3.3 The Shapley Value as a Utility Function

Until now I have considered a cooperative game strictly as a description of potentially attainable values by coalitions in some cooperative interactive decision situation. Then we proceeded to ask the fundamental question of cooperative game theory: "How do we allocate the wealth generated over the various individuals in the

[5] As van den Brink (2001) reports, the normalized Banzhaf value satisfies efficiency as well as the null-player property, but is not fair. For a definition of the normalized Banzhaf value I refer to van den Brink (2001).

situation?" These allocations are mainly determined by negotiating power (the Core) or combination of attainability and fairness considerations (the Shapley value).

Here we develop a very different perspective on value theory. Roth introduced a utility perspective on the Shapley value. His theory is developed in the seminal contributions Roth (1977a, 1977b) and further discussed in Roth (1988). Roth's perspective is that a value represents a valuation or utility of a player regarding a cooperative interactive decision situation in which this player participates. Hence, a cooperative game describes the potential values that can be generated by the various coalitions of which the player is a member. The player evaluates this situation using a utility function. A value—in particular, the Shapley value—could act as such a utility function.

Roth found that the Shapley value can be interpreted as such a utility function with very interesting properties. In fact he showed that the Shapley value is a vNM utility function satisfying several risk neutrality hypotheses. I first summarize the general concept of a vNM utility function, before proceeding to investigate the Shapley value from this perspective.

As a preliminary, consider an arbitrary set X. A *binary relation* on X is defined as a subset R of the collection of all ordered pairs in X, i.e., $R \subset X \times X$. An ordered pair $(x, y) \in R$ is usually denoted by xRy.

A binary relation R on X is *complete* if for every pair $x, y \in X$ it holds that xRy or yRx. Furthermore, a binary relation R on X is *transitive* if for every triple $x, y, z \in X$ it holds that xRy and yRz imply that xRz. A binary relation R on X is denoted as a *preference relation* or simply as a "preference" if R is complete as well as transitive.

Now I turn to the development of Roth's approach to the Shapley value. Throughout we let $\mathbb{L}$ be a collection of *lotteries*, where a lottery is defined as a probability distribution ℓ on some finite set of events E with $|E| < \infty$. Thus, $\ell \colon E \to [0,1]$ such that $\sum_{e \in E} \ell(e) = 1$.

If $\ell_1, \ell_2 \in \mathbb{L}$ and $p \in [0,1]$, then the compound lottery is given by $[p\,\ell_1; (1-p)\,\ell_2] = p\,\ell_1 + (1-p)\,\ell_2 \in \mathbb{L}$.

Definition 3.17 Let $\succcurlyeq$ be some preference relation on the class of lotteries $\mathbb{L}$. Then a function $u \colon \mathbb{L} \to \mathbb{R}$ is an *expected utility function* for preference $\succcurlyeq$ on $\mathbb{L}$ if

$$\ell_1 \succcurlyeq \ell_2 \text{ if and only if } u(\ell_1) \geqq u(\ell_2) \tag{3.19}$$

and

$$u\,(p\,\ell_1 + (1-p)\,\ell_2) = p\,u(\ell_1) + (1-p)\,u(\ell_2) \tag{3.20}$$

An expected utility function represents the preference over a lottery by taking the expected utility resulting from some evaluation of the underlying events. This evaluation of the underlying events is usually denoted as the vNM utility function that represents the preference relation under consideration.[6] The main problem in vNM utility theory is the existence of such expected utility representations of preference relation on collections of lotteries.

Before we proceed, it is necessary to introduce some auxiliary notation. Let $\ell_1, \ell_2 \in \mathbb{L}$. Now $\ell_1 \sim \ell_2$ if and only if $\ell_1 \succcurlyeq \ell_2$ as well as $\ell_2 \succcurlyeq \ell_1$. Also, $\ell_1 \succ \ell_2$ if and only if $\ell_1 \succcurlyeq \ell_2$ and not $\ell_2 \succcurlyeq \ell_1$.

The following lemma summarizes the main insight from expected utility theory on the existence of vNM expected utility functions. We give this insight without proof.

Lemma 3.18 *Let $\succcurlyeq$ be given on $\mathbb{L}$. There exists an expected utility function u for $\succcurlyeq$ on $\mathbb{L}$ if the preference relation $\succcurlyeq$ satisfies the two following conditions:*

Continuity *For all lotteries $\ell_1, \ell_2 \in \mathbb{L}$, both the sets $\{p \mid [p\,\ell_1;(1-p)\,\ell_2] \succcurlyeq \ell\}$ and $\{p \mid [p\,\ell_1;(1-p)\,\ell_2] \preccurlyeq \ell\}$ are closed for all $\ell \in \mathbb{L}$.*
Substitutability *If $\ell_1 \sim \ell_2$, then for any $\ell \in \mathbb{L}$*

$$\left[\tfrac{1}{2}\,\ell_1;\tfrac{1}{2}\,\ell\right] \sim \left[\tfrac{1}{2}\,\ell_2;\tfrac{1}{2}\,\ell\right].$$

Furthermore, the expected utility function u is unique up to affine transformations.

Next in the development of Roth's theory of the Shapley value, I turn to the modeling of preferences over so-called "game positions" as probabilistic events rather than deterministic constructions. For that we define $\mathbb{E} = N \times \mathcal{G}^N$ as the collection of game positions. Here, we interpret a pair $(i, v) \in \mathbb{E}$ as the position of player i in cooperative game v. Or, for short, (i, v) is a *game position*. One can now consider a game position to be a probabilistic event: The fact that one has the position of player i in game v is an outcome of a complex process that is subject to many probabilistic influences.

Now I consider the preference $\succcurlyeq$ on $\mathbb{E}$. Here, $(i, v) \succcurlyeq (j, w)$ means that it is preferable to have position i in game v over having position j in game w. These preferences are assumed to be subjective, i.e., these preferences are individualistic. This latter assumption distinguishes this theory from standard value theory.

We also consider probabilistic mixtures of positions. Here $[p\,(i, v);(1-p)\,(j, w)]$ means that one has a game position (i, v) with probability p and game position (j, w) with probability $1-p$. Now we re-define $\mathbb{L}$ as the set of all lotteries over $\mathbb{E}$. Hence, the set $\mathbb{L}$ contains exactly all lotteries over game positions. Clearly the preference relation $\succcurlyeq$ can be applied to $\mathbb{L}$ with the method described above.

Modeling Hypothesis 3.19 *The preference $\succcurlyeq$ over the space of lotteries over game positions $\mathbb{L}$ satisfies continuity and substitutability.*

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[6] It is interesting to note here that von Neumann and Morgenstern (1953) provided the first treatment of expected utility functions. von Neumann and Morgenstern were the first to consider evaluating lotteries by the expected utility that can be obtained based on some utility function on the underlying events. In their honor we talk about vNM utility functions ever since.

From the hypothesis above we can immediately conclude that there exists some vNM expected utility function $\psi : \mathbb{E} \to \mathbb{R}$ such that for every lottery $\ell \in \mathbb{L}$:

$$\psi(\ell) = \sum_{(i,v)\in\mathbb{E}} \ell(i,v) \cdot \psi_i(v), \tag{3.21}$$

where $\psi_i(v) = \psi(i,v)$.

Regarding the preference relation $\succcurlyeq$ we can make certain regularity assumptions. I will develop these hypotheses sequentially and state some intermediary results as we discuss these hypotheses.

Axiom 3.20 *Let $\mathcal{G}^N_{-i}$ be the class of games in which player i is a null player and let η be the null game as introduced in Chapter 1 with $\eta(S) = 0$ for all $S \subset N$.*

If $v \in \mathcal{G}^N_{-i}$, then $(i,v) \sim (i,\eta)$.

Axiom 3.20 states that a player is indifferent between being a null player in an arbitrary game and having a game position in the null game η itself. Both positions are equally undesirable for the player under consideration.

Axiom 3.21 *For any permutation $\rho : N \rightleftarrows N$ it holds that $(i,v) \sim (\rho(i), \rho\, v)$.*

Axiom 3.21 is an anonymity hypothesis: The name of a position is irrelevant for the evaluation of the player's position in the game. In other words, the evaluation of a game position is only based on the values generated rather than personal information.

From Axioms 3.20 and 3.21 we derive two properties of the vNM expected utility function ψ:

(a) The expected utility function is symmetric in the sense that for any permutation ρ we have that $\psi_i(v) = \psi_{\rho(i)}(\rho\, v)$.
(b) We can normalize ψ by choosing $\psi_i(u_{\{i\}}) = 1$ and $\psi_i(\eta) = 0$.

The third axiom that we introduce links scaling of payoffs and the introduction of risk. Indeed, it states that scaling the rewards in a game is equivalent to the introduction of risk.

Axiom 3.22 *For any number $C > 1$ and game position $(i,v) \in \mathbb{E}$ it holds that*

$$(i,v) \sim \left[\tfrac{1}{C}\,(i, C\,v); \left(1 - \tfrac{1}{C}\right)\,(i,\eta) \right] \tag{3.22}$$

From the three axioms introduced thus far we can derive another familiar property. Namely, it states that the three axioms imply that the expected utility function is linear.

Lemma 3.23 *Under Axioms 3.20, 3.21 and 3.22 it holds that for every $C \geqq 0$ and game position $(i, v) \in \mathbb{E}$: $\psi_i(C\,v) = C\,\psi_i(v)$.*

Proof If $C = 0$ then by Axiom 3.20 it follows that $\psi_i(C\,v) = \psi_i(\eta) = 0$. Hence, we only investigate $C > 0$.

Without loss of generality we may assume that $C \geqq 1$.[7] Then by Axiom 3.22 we may apply (3.22) to arrive at

$$\begin{aligned}\psi_i(v) &= \psi_i\left[\tfrac{1}{C}\,(i, C\,v); \left(1 - \tfrac{1}{C}\right)\,(i, \eta)\right] = \\ &= \tfrac{1}{C}\,\psi_i(C\,v) + \left(1 - \tfrac{1}{C}\right)\,\psi_i(\eta) = \\ &= \tfrac{1}{C}\,\psi_i(C\,v).\end{aligned}$$

This shows the assertion. ■

Axiom 3.24 (Ordinary risk neutrality) *For every player i we have that*

$$(\,i, (q\,w + (1 - q)\,v)\,) \sim \left[\,q(i, w);(1 - q)(i, v)\,\right],$$

where $v, w \in \mathcal{G}^N$ and $q \in [0, 1]$.

Axiom 3.24 introduces additional properties into the discussion. The following properties reflect the consequences of this hypothesis for the expected utility function ψ.

Proposition 3.25 *The following two properties hold.*

(a) *Axiom 3.24 implies Axiom 3.22.*
(b) *Under Axioms 3.20, 3.21 and 3.22 it holds that $\succcurlyeq$ satisfies Axiom 3.24 if and only if for all games $v, w \in \mathcal{G}^N$ and every player $i \in N$:*

$$\psi_i(v + w) = \psi_i(v) + \psi_i(w)$$

Proof Assertion (a) is rather trivial. Therefore, we limit ourselves to the proof of assertion (b).

For all games $v, w \in \mathcal{G}^N$ and every player $i \in N$ we have

$$\psi_i(v + w) = \psi_i\left(2\left(\tfrac{1}{2}\,v + \tfrac{1}{2}\,w\right)\right) = 2\,\psi_i\left(\tfrac{1}{2}\,v + \tfrac{1}{2}\,w\right)$$

which follows from the axioms formulated. Now by Axiom 3.24 it follows that

$$\psi_i\left(\tfrac{1}{2}\,v + \tfrac{1}{2}\,w\right) = \psi\,\left[\,\tfrac{1}{2}\,(i, v);\tfrac{1}{2}\,(i, w)\right] = \tfrac{1}{2}\,\psi_i(v) + \tfrac{1}{2}\,\psi_i(w)$$

[7] Indeed, for $0 < C < 1$ we can select $C' = \frac{1}{C}$.

using the expected utility formulation. ■

Assertion (b) in Proposition 3.25 links ordinary risk neutrality to additivity. It is clear that this in conjunction with Axioms 3.20, 3.21 and 3.22 brings us a step closer to characterizing the Shapley value as a vNM expected utility function on the class of game positions.

Until now we limited our discussion to some rather standard axioms of expected utility theory. The only exception being the neutrality with respect to ordinary risk (Axiom 3.24). Now we turn to the introduction of one additional essential axiom of risk neutrality, namely the neutrality with respect to *strategic* risk.

We first note that the unanimity game u_S for $S \subset N$ can be interpreted as a *pure bargaining game* in its simplest form. Indeed, it essentially describes the strategic bargaining of $|S|$ players over one unit of wealth. This leads to the following concept.

Definition 3.26 The *certainty equivalent* of a strategic position in the unanimity game u_S is a number $\bar{\gamma}(s)$, where $s = |S|$, such that for every position $i \in S$:

$$(i, u_S) \sim \left(i, \bar{\gamma}(s)\, u_{\{i\}} \right) \tag{3.23}$$

Remark that $\bar{\gamma}(1) = 1$ and that $\bar{\gamma}(s)$ is a measure of a player's opinion of her own bargaining ability in a pure bargaining game of the size s.

Under standard assumptions such as continuity of the preference relation, one can show that certainty equivalents exist. Here, we assume that there is no problem with the existence of these certainty equivalents and we can use this concept in the definition of certain properties. With reference to Roth (1977a) we introduce the following terminology:

- The preference relation $\succcurlyeq$ reflects an *aversion to strategic risk* if for all $s = 1, \ldots, n$ it holds that $\bar{\gamma}(s) \leqq \frac{1}{s}$.
- The preference $\succcurlyeq$ is *neutral to strategic risk* if for all $s = 1, \ldots, n$ it holds that $\bar{\gamma}(s) = \frac{1}{s}$.
- The preference $\succcurlyeq$ is *strategic risk loving* if for all $s = 1, \ldots, n$ it holds that $\bar{\gamma}(s) \geqq \frac{1}{s}$.

The following result, which was seminally formulated in Roth (1977a), is given without a detailed proof. It follows quite straightforwardly from the results already formulated and discussed.

Theorem 3.27 *The Shapley value φ is the unique expected utility function on $\mathbb{E} = N \times \mathcal{G}^N$ that satisfies Axioms 3.20, 3.21, 3.22 and 3.24 and is neutral to strategic risk.*

The theorem states that the Shapley value is a very specific vNM expected utility function. It is exactly the one that is neutral to ordinary risk as well as neutral to strategic risk. In this respect, therefore, the Shapley value is neutral to the main forms of risk.

Beyond the realm of vNM utility theory, this insight is useful, since it provides a foundation for the use of the Shapley value as a utilitarian benchmark in the analysis of situations in which cooperative games describe certain value-generating situations. This is exactly the case for the analysis of the exercise of authority in hierarchical firms, developed by van den Brink and Gilles (2009). There a firm is a combination of an authority hierarchy and a production process. The first is described by a hierarchical network of authority relations and the second by a cooperative game that assigns to every coalition of productive workers a team production value. Every worker is now assumed to have a preference over the various positions that he can assume in this firm. Application of the Shapley value now exactly applies a risk neutral evaluation of these positions. It, thus, functions as a benchmark in this analysis.

3.4 The Myerson Value

Myerson (1977) and Myerson (1980) introduced a generalization of the Shapley value for decision situations with constraints on coalition formation. Myerson (1977) originally introduced network-based constraints on coalition formation in which context he discussed the resulting Shapley value.[8] In this section I provide an overview of how more general constraints on coalition formation affect the resulting Shapley value.

With reference to the discussion in Chapter 2, I recall that $\Omega \subset 2^N$ is an (institutional) coalitional structure on N if $\varnothing \in \Omega$. As in my discussion of the restricted Core $\mathcal{C}(\Omega, v)$ of a cooperative game $v \in \mathcal{G}^N$ in Chapter 2, I now turn to the question whether we can define an axiomatic value for such a situation in which there are constraints on coalition formation. Furthermore, I require this value to be closely related to the Shapley value in the sense that for the case without restrictions on coalition formation it is equivalent to the Shapley value. Myerson introduced the most plausible extension of the Shapley value for such situations, called the *Myerson value*.

The key assumption of the Myerson value is that only institutional coalitions can truly generate dividends that should be divided equally among the members of these coalitions. The definition of the Myerson value is constructed for the most general case in the next definition. There are some technical problems with defining the appropriate institutional coalitional structures for that. Next, we first introduce the technical preliminaries from Algaba, Bilbao, Borm, and López (2001) for the definition of the Myerson value.

Let $\Omega \subset 2^N$ such that $\varnothing \in \Omega$. We define the following concepts based on Ω:

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[8] Extensions of Myerson's analysis were pursued by Jackson and Wolinsky (1996), who extended the Myerson value to arbitrary network situations. For an elaborate discussion of network-based constraints on coalition formation in this sense I also refer to Jackson (2008, Chapter 12).

A basis of coalitional structure Ω. We define the collection of *supportable coalitions* in the coalitional structure Ω by

$$N(\Omega) = \{S \in \Omega \mid \exists\, T, R \in \Omega \setminus \{\varnothing, S\} \colon\ S = T \cup R \text{ and } T \cap R \neq \varnothing\} \quad (3.24)$$

The *basis* of Ω is now defined by

$$B(\Omega) = \Omega \setminus N(\Omega). \quad (3.25)$$

Hence, the basis of Ω consists exactly of the non-supportable coalitions. For the case that $\Omega = 2^N$ is the collection of *all* potential coalitions, it is clear that $B(2^N) = \{\, \{i, j\} \mid i \neq j \text{ and } i, j \in N\} \cup \{\, \{i\} \mid i \in N\} \cup \{\varnothing\}$.

Following (Algaba et al. 2001) the coalitions in $B(\Omega)$ are called the *supports* of Ω in the sense that every institutional coalition $S \in \Omega$ can be written as a union of supports in $B(\Omega)$.

The union stable cover of Ω. The *union stable cover* of the coalitional structure Ω is the smallest collection $\overline{\Omega} \subset 2^N$ such that $\Omega \subset \overline{\Omega}$ and for all $S, T \in \overline{\Omega}$ with $S \cap T \neq \varnothing$ it holds that $S \cup T \in \overline{\Omega}$.

It is easy to see that the union stable cover $\overline{\Omega}$ of Ω is equal to the union stable cover $\overline{B(\Omega)}$ of its basis $B(\Omega)$.

Ω-*Components.* Let $S \subset N$ be an arbitrary coalition. The family of Ω-*components* of S is defined by

$$C_\Omega(S) = \left\{ T \in \overline{\Omega} \mid T \subset S \text{ and there is no } R \in \Omega \colon\ T \subsetneq R \subset S \right\} \quad (3.26)$$

A coalition $S \in C_\Omega(N)$ is called a *global* component here if it forms an Ω-component of the grand coalition N. We remark here that following Lemma 3.28 below the global components in the player set N are pairwise disjoint.

The grand Ω-*component.* The *grand* Ω-component is defined by

$$\widehat{C}_\Omega = \cup C_\Omega(N) \subset N \quad (3.27)$$

Hence, the grand Ω-component is the union of all global Ω-components of the grand coalition N. Players $i \notin \widehat{C}_\Omega$ that are not a member of the global Ω-component are denoted as Ω-*isolated* players—or simply as isolated players if the context is clear. These isolated players are not really involved in the building of any institutional or formable coalition $S \in \Omega$.

We require certain regularity conditions to be satisfied to define the Myerson value for a game with a coalitional structure. Crucial is the following property that $C_\Omega(S)$ is a collection of pairwise disjoint coalitions.

Lemma 3.28 *For every coalition $S \subset N$ with $C_\Omega(S) \neq \varnothing$, the Ω-components of S form a collection of pairwise disjoint subcoalitions of S.*

Proof Let $S \subset N$ be such that $C_\Omega(S) \neq \varnothing$. Assume that $T, R \in \overline{\Omega}$ are two components of S. If $T \cap R \neq \varnothing$, then $T \cup R \in \overline{\Omega}$. Hence, $T \cup R \in C_\Omega(S)$, which contradicts the maximality of T and R. Hence, $T \cap R = \varnothing$. ∎

The previous discussion of these concepts allows us to define the Myerson restriction of a cooperative game and allow the introduction of the Myerson value.

Definition 3.29 Let $\Omega \subset 2^N$ be a coalitional structure such that $\varnothing \in \Omega$ and let $v \in \mathcal{G}^N$.

(a) The Ω-*restriction* of the game v is given by $v_\Omega \colon 2^N \to \mathbb{R}$ with

$$v_\Omega(S) = \sum_{T \in C_\Omega(S)} v(T). \tag{3.28}$$

(b) Let $\mathfrak{W}^N = \{\Omega \mid \Omega \subset 2^N\}$ be the collection of all coalitional structures on N. The *Myerson value* is a function $\mu \colon \mathcal{G}^N \times \mathfrak{W}^N \to \mathbb{R}^N$ such that for every game v and every player $i \in N$ the Myerson value is the Shapley value of its Ω-restriction, i.e.,

$$\mu_i(v, \Omega) = \varphi_i\left(v_\Omega\right). \tag{3.29}$$

The Ω-restriction v_Ω of an arbitrary game $v \in \mathcal{G}^N$ is similar to the Kaneko–Wooders partitioning game introduced in the previous chapter. Indeed, if Ω is a partitioning of N, the two notions are the same. In the current context, however, the restricted game is based on the restricted game as introduced for communication network situations in Myerson (1977) and for conference structures in Myerson (1980).

The Myerson value is usually characterized by three axioms. Two of these axioms—component efficiency and the balanced payoff property[9]—have seminally been introduced by Myerson (1977). In this case these axioms are generalized to the setting of arbitrary coalitional structures rather than the coalitional generated by a communication network discussed in the seminal work by Myerson.

Let $\phi \colon \mathcal{G}^N \times \mathfrak{W}^N \to \mathbb{R}^N$ be some allocation rule on the class of games with a coalitional structure. Then we introduce three fundamental properties:

Component-efficiency property. For every global component $S \in C_\Omega(N)$:

$$\sum_{i \in S} \phi_i(v, \Omega) = v(S). \tag{3.30}$$

[9] The balanced payoff property was introduced under the nomen "fairness" in Myerson's seminal discussion of value allocation in a communication network. Later this was recognized as the wrong indication; Jackson modified the nomenclature to the balanced payoff property.

Isolated player property. For every Ω-isolated player $i \in N \setminus \widehat{C}_\Omega$ and for every $v \in \mathcal{G}^N$ it holds that $\mu_i(v, \Omega) = 0$.

Balanced payoff property. For every support coalition $S \in B(\Omega)$ and for all players $i, j \in S$:

$$\phi_i(v, \Omega) - \phi_i(v, \Omega') = \phi_j(v, \Omega) - \phi_j(v, \Omega') \tag{3.31}$$

where $\Omega' = \overline{B(\Omega) \setminus \{S\}}$ the union stable cover of the coalitional structure excluding support S.

The following theorem generalizes the results stated in Myerson (1977) and Myerson (1980). The result as stated below was developed in Algaba et al. (2001).

Theorem 3.30 *The Myerson value μ is the unique allocation rule on the class of games with a coalitional structure $\mathcal{G}^N \times \mathfrak{W}^N$ that satisfies the component-efficiency, isolated player and balanced payoff properties.*

For a proof of Theorem 3.30 I refer to the appendix of this chapter.

3.5 Appendix: Proofs of the Main Theorems

Proof of Theorem 3.7

Proof of (3.3)
Implement the definition of the Harsanyi dividend for any coalition S in the game v, given by

$$\Delta_v(S) = \sum_{T \subset S} (-1)^{|S|-|T|} v(T),$$

into the definition of the Shapley value:

$$\varphi_i(v) = \sum_{S \subset N:\, i \in S} \frac{1}{|S|} \left[\sum_{T \subset S} (-1)^{|S|-|T|} v(T) \right] = \sum_{T \subset N} \left[\sum_{S \subset N:\, T \cup \{i\} \subset S} (-1)^{|S|-|T|} \frac{v(T)}{|S|} \right]$$

Write

$$\delta_i(T) = \sum_{S \subset N:\, T \cup \{i\} \subset S} (-1)^{|S|-|T|} \frac{1}{|S|} \tag{3.32}$$

It is easy to see that if $i \notin T'$ and $T = T' \cup \{i\}$, then $\delta_i(T) = -\delta_i(T')$. All the terms on the right hand side of (3.32) are the same in both cases, except that $|T| = |T'| + 1$. Thus, a sign change can be applied throughout to arrive at

$$\varphi_i(v) = \sum_{T \subset N:\, i \in T} \delta_i(T) \left[v(T) - v(T \setminus \{i\}) \right]$$

Now if $i \in T$ and T has t members, there are exactly $\binom{n-t}{s-t}$ coalitions S with s members such that $T \subset S$. Thus,

$$\begin{aligned}
\delta_i(T) = \sum_{s=t}^{n} (-1)^{s-t} \binom{n-t}{s-t} \frac{1}{s} &= \sum_{s=t}^{n} (-1)^{s-t} \binom{n-t}{s-t} \int_0^1 x^{s-1}\, dx = \\
&= \int_0^1 \sum_{s=t}^{n} (-1)^{s-t} \binom{n-t}{s-t} x^{s-1}\, dx \\
&= \int_0^1 x^{t-1} \sum_{s=t}^{n} (-1)^{s-t} \binom{n-t}{s-t} x^{s-t}\, dx = \int_0^1 x^{t-1} (1-x)^{n-t}\, dx.
\end{aligned}$$

This is a well-known integral, leading to

$$\delta_i(T) = \frac{(t-1)!\,(n-t)!}{n!}.$$

This implies (3.3) as required.

Proof of (3.4)
This is the direct consequence of (3.3) and the definition of Weber strings and marginal values based on these Weber strings. The proof is therefore omitted.

Proof of (3.5)
Recall that $E_v(x) = \sum_{S \subset N} \Delta_v(S) \prod_{i \in S} x_i$. This implies that

$$\frac{\partial E_v}{\partial x_i}(x) = \sum_{S \subset N:\, i \in S} \Delta_v(S) \prod_{j \in S \setminus \{i\}} x_j$$

and, so,

$$\frac{\partial E_v}{\partial x_i}(t, \ldots, t) = \sum_{S \subset N:\, i \in S} \Delta_v(S) t^{|S|-1}.$$

This in turn implies that

$$\begin{aligned}
\int_0^1 \frac{\partial E_v}{\partial x_i}(t, \ldots, t)\, dt = \int_0^1 \sum_{S \subset N:\, i \in S} \Delta_v(S) t^{|S|-1}\, dt = \\
= \sum_{S \subset N:\, i \in S} \frac{\Delta_v(S)}{|S|} t^{|S|} \Big|_0^1 = \sum_{S \subset N:\, i \in S} \frac{\Delta_v(S)}{|S|} = \varphi_i(v)
\end{aligned}$$

This shows (3.5).

Proof of Theorem 3.13

Consider an arbitrary value ϕ on $\mathcal{G}^N$.

First we show that strong monotonicity implies the null-player property. Indeed, note that if ϕ is strongly monotone then for all games $v, w \in \mathcal{G}^N$:

$$D_i v(S) = D_i w(S) \text{ for all } S \subset N \text{ implies } \phi_i(v) = \phi_i(w). \tag{3.33}$$

Next consider the null game η defined by $\eta(S) = 0$ for all $S \subset N$. Then by symmetry we have that $\phi_i(\eta) = \phi_j(\eta)$ for all $i, j \in N$. By efficiency $\sum_N \phi_i(\eta) = \eta(\mathrm{N})=0$, and therefore $\phi_i(\eta) = 0$ for all $i \in N$.

Let $v \in \mathcal{G}^N$. Now by (3.33) it follows for any $i \in N$:

$$D_i v(S) = 0 = D_i \eta(S) \text{ for all } S \subset N \text{ implies } \phi_i(v) = 0 \tag{3.34}$$

Hence we have show the null-player property.

Again write the game v in its unanimity basis form $v = \sum_S \Delta_v(S)\, u_S$. Define the index I as the minimum number of non-zero terms in the unanimity decomposition of the game v. We prove the main assertion by induction on the index I.

$I = 0$ Then every player is a null-player, namely $v = \eta$. Hence, $\phi_i(v) = \phi_i(\eta) = 0 = \varphi_i(v)$.

$I = 1$ Then $v = C\, u_S$ for some $S \subset N$. For $i \notin S$ we have that $D_i(T) = 0$ for all $T \subset N$, and, thus, by (3.34), $\phi_i(v) = 0$. For all $i, j \in S$ symmetry implies that $\phi_i(v) = \phi_j(v)$. Combined with efficiency this implies that $\phi_i(v) = \frac{C}{|S|} = \varphi_i(v)$ for all $i \in S$.

Assume the assertion holds for I; prove it holds for $I+1$. Assume that $\phi(w) = \varphi(w)$ for any game w with an index less or equal to I. Let v have index $I+1$. Then we can write

$$v = \sum_{k=1}^{I+1} \Delta_v(S_k)\, u_{S_k} \text{ with } \Delta_v(S_k) \neq 0.$$

Define $S = \cap_{k=1}^{I+1} S_k$ and suppose that $i \notin S$. Define the game

$$w = \sum_{k:\ i \in S_k} \Delta_v(S_k)\, u_{S_k}$$

Then w has an index of at most I with regard to $i \notin S$. Furthermore, $D_i w(T) = D_i v(T)$ for all $T \subset N$. Thus, by the induction hypothesis and strong monotonicity, it can be concluded that

$$\phi_i(v) = \phi_i(w) = \sum_{k:\ i \in S_k} \frac{\Delta_v(S_k)}{|S_k|} = \varphi_i(v)$$

Next suppose that $i \in S$. By symmetry, $\phi_i(v)$ is a constant c for all players in S. Likewise the Shapley value $\varphi_i(v)$ is a constant c' for all members of S. Since both values ϕ and φ satisfy efficiency and are equal for all i not in S, it has to be concluded that $c = c'$.

This completes the proof of Theorem 3.13.

Proof of Theorem 3.16

The proof of van den Brink's theorem is rather involved. The proof is based on the application of graph-theoretic concepts and techniques.

Suppose that ϕ satisfies efficiency, the null-player property and fairness. For every $v \in \mathcal{G}^N$ define its support by

$$S(v) = \{S \subset N \mid \Delta_v(S) \neq 0\}$$

and $I(v) = |S(v)|$ as the number of coalitions in its support, also denoted as the index of the game v.[10] As in the proof of Young's axiomatization we apply induction on the index $I(v)$.

$$I(v) = 0, \text{i.e.}, v = \eta.$$

In this case the null-player property implies that $\phi_i(v) = 0 = \varphi_i(v)$.

$$I(v) = 1, \text{i.e.}, v = C\, u_S \text{for some} S \subset N.$$

Since the null-player property and fairness imply symmetry, we can apply the same reasoning as the one used in the proof of Theorem 3.13. Hence we immediately conclude that

$$\phi_i(C\, u_S) = \varphi_i(C\, u_S) = \frac{C}{|S|}.$$

Proceed by induction: Assume that $\phi(v') = \varphi(v')$ for all games v' with $I(v') \leqq I$.

Now let $v \in \mathcal{G}^N$ be such that $I(v) = I + 1 \geqq 2$. Now we introduce a graph $g_v \subset \{ij \mid i,j \in N\}$ on N, where $ij = \{i,j\}$ represents an undirected link between i and j.

In particular, define $ij \in g_v$ if and only if $i \neq j$ and there exists some coalition $S \in S(v)$ with either $ij \subset S$ or $S \cap ij = \varnothing$.

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[10] Here I also refer to the proof of Theorem 3.13. There this index was simply introduced as I.

A *component* of g_v is a coalition $S \subset N$ that is maximally connected[11] in g_v, i.e., S is connected in g_v and it does not contain a strict superset that is connected in g_v as well.

We distinguish two cases for the graph g_v:

Case A: *N is a component of g_v.*

Now define

$$N'(v) = \{i \in N \mid i \in T \text{ for some } T \in S(v)\} \tag{3.35}$$

as the set of non-null players in v. Take $j \in N'(v)$ and let $T_0 = \{j\}$. For k we recursively define

$$T_k = \left\{ i \in N \setminus \left(\cup_{m=0}^{k-1} T_m \right) \;\middle|\; \text{There exists some } j \in T_{k-1} \text{ with } ij \in g_v \right\} \tag{3.36}$$

The collection $T = (T_0, \dots, T_K)$ is a partition of N consisting on non-empty coalitions only. Indeed, since N is a component of g_v the given procedure yields non-empty coalitions T_k.

Recall that $T_0 = \{j\}$. Now suppose that $\phi_j(v) = F$ for some $F \in \mathbb{R}$ and define $c_j = 0$. We now recursively will determine $\phi_i(v)$ for all $i \in T_k$ with $k = 1, \dots, m$.

Indeed, for $i \in T_k$ with $k \in \{1, \dots, m\}$ there exists some $h \in T_{k-1}$ and $T \in S(v)$ with either $ih \subset T$ or $T \cap ih = \varnothing$. Fairness now implies that

$$\phi_i(v) - \phi_i(v - \Delta_v(T)\, u_T) = \phi_h(v) - \phi_h(v - \Delta_v(T)\, u_T)$$

But then

$$\begin{aligned} \phi_i(v) &= \phi_h(v) - \phi_h(v - \Delta_v(T)\, u_T) + \phi_i(v - \Delta_v(T)\, u_T) = \\ &= F + c_h - \phi_h(v - \Delta_v(T)\, u_T) + \phi_i(v - \Delta_v(T)\, u_T) = F + c_i, \end{aligned} \tag{3.37}$$

where $c_i = c_h - \phi_h(v - \Delta_v(T)\, u_T) + \phi_i(v - \Delta_v(T)\, u_T)$. The value of c_i is determined since by the induction hypothesis both $\phi_h(v - \Delta_v(T)\, u_T)$ and $\phi_i(v - \Delta_v(T)\, u_T)$ are fully determined.

Efficiency now implies that

$$\sum_{i \in N} \phi_i(v) = n \cdot F + \sum_{i \in N} c_i = v(N).$$

From this it follows that $F = \frac{1}{n} \left(v(N) - \sum_N c_i \right)$ is uniquely determined. With $\phi_j(v) = F$ and (3.37) it can be concluded that all values $\phi_i(v)$, $i \in N$ are fully determined.

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[11] Coalition S is *connected* in g_v if for all $i, j \in S$ with $i \neq j$ there exists a sequence $i_1, \dots, i_K \in S$ with $i_1 = i$, $i_K = j$ and $i_k i_{k+1} \in g_v$ for all $k = 1, \dots, K-1$.

Case B: N is not a component of g_v.
Case B is equivalent to the property that g_v is not a connected graph or network. Thus, there exist at least two components in g_v, denoted by B_1 and B_2 with $B_1 \cap B_2 = \varnothing$.[12]

We first show that g_v can at most have two components. Suppose to the contrary that there is a third component, B_3. Since $I(v) \geqq 2$ we have that $S(v) \neq \varnothing$. Assume without loss of generality that there exists $T \in S(v)$ with $T \subset B_3$. If $i \in B_!$ and $j \in B_2$, then $ij \cap R = \varnothing$. But then by definition $ij \in g_v$, which would be a contradiction to B_1 and B_2 being components of g_v.

So, g_v has exactly two components, say $V = B_1$ and $W = B_2$. Since $I(v) \geqq 2$ v can be written as the sum of two unanimity games, i.e., $v = C_V\, u_V + C_W\, u_W$ with $C_V, C_W \neq 0$. Also, $V \cup W = N$.[13]

We can now distinguish two cases regarding the size of N:

1. Suppose $n \geqq 3$.
   Suppose without loss of generality that $|V| \geqq 2$. Take $j \in V$ and $h \in W$. Define $w \in \mathcal{G}^N$ by[14]

$$w = v + C_V\, u_{V-j+h} = C_V\, u_V + C_V\, u_{V-j+h} + C_W\, u_W.$$

   We first determine $\phi(w)$. Let $\phi_h(w) = F$. Fairness now implies that $\phi_j(w) - \phi_j(c_W\, u_W) = \phi_h(w) - \phi_h(C_W\, u_W)$. Since $I(C_W\, u_W) = 1$, we already determined that $\phi_j(C_W\, u_W) = 0$ and $\phi_h(C_W\, u_W) = \frac{C_W}{|W|}$. So,

$$\phi_j(w) = \phi_h(w) - \phi_h(C_W\, u_W) + \phi_h(C_W\, u_W) = F - \tfrac{C_W}{|W|}.$$

   The null-player property implies that $\phi_i(C_V\, u_V + C_V\, u_{V-j+h}) = 0$ for $i \in W \setminus \{h\}$. Fairness and the fact that $I(C_V\, u_{V-j+h}) = 1$ implies for $i \in W \setminus \{h\}$ that

$$\begin{aligned}\phi_h(C_V\, u_V + C_V\, u_{V-j+h}) &= \phi_i(C_V\, u_V + C_V\, u_{V-j+h}) - \phi_i(C_V\, u_{V-j+h}) + \\ &+ \phi_h(C_V\, u_{V-j+h}) = \tfrac{C_V}{|V|}.\end{aligned}$$

   For every $i \in W \setminus \{h\}$, fairness then implies that

$$\phi_i(w) = \phi_h(w) - \phi_h(C_V\, u_V + C_V\, u_{T-j+h}) + \phi_i(C_V\, u_V + C_V\, u_{T-j+h}) = F - \tfrac{C_V}{|V|}.$$

   For $i \in V \setminus \{j\}$ fairness in a similar fashion implies that

$$\phi_i(w) = \phi_j(w) - \phi_j(C_V\, u_{V-j+h}) + \phi_i(C_V\, u_{V-j+h}) = F - \tfrac{C_W}{|W|} + \tfrac{C_V}{|V|}.$$

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[12] It follows from this as well that $n \geqq 2$.

[13] If $V \cup W \neq N$, then for $i \in V, j \in W$ and $h \in N \setminus (V \cup W)$ it holds that $ih, jh \in g_v$. This is a contradiction to V and W being components of g_v.

[14] Here I use the notation $V - j + h = (V \setminus \{j\}) \cup \{h\}$.

We can summarize the conclusions now as follows:

$$\phi_i(w) = \begin{cases} F & \text{if } i = h \\ F - \frac{C_W}{|W|} & \text{if } i = j \\ F - \frac{C_V}{|V|} & \text{if } i \in W \setminus \{h\} \\ F - \frac{C_W}{|W|} + \frac{C_V}{|V|} & \text{if } i \in V \setminus \{j\} \end{cases} \tag{3.38}$$

With efficiency it now has to hold that

$$\sum_{i \in N} \phi_i(w) = n\,F - \frac{|V|}{|W|} C_W + \frac{|V| - |W|}{|V|} C_V \equiv 2C_V + C_W.$$

This in turn implies that

$$F = \frac{2|V| - |V| + |W|}{|V| \cdot n} C_V + \frac{|W| + |V|}{|W| \cdot n} C_W = \tfrac{C_V}{|V|} + \tfrac{C_W}{|W|}.$$

With the above this then determines that

$$\phi_i(w) = \begin{cases} \frac{C_V}{|V|} + \frac{C_W}{|W|} & \text{if } i = h \\ \frac{C_V}{|V|} & \text{if } i = j \\ \frac{C_W}{|W|} & \text{if } i \in W \setminus \{h\} \\ 2\frac{C_V}{|V|} & \text{if } i \in V \setminus \{j\} \end{cases} \tag{3.39}$$

We can finally determine the values $\phi_i(v)$, $i \in N$. Let $\phi_h(v) = F'$ be the anchor value.

Fairness and the null-player property together with Proposition 3.15(c) implies that $\phi_i(v) = F'$ for all $i \in W$.
For every $i \in V \setminus \{j\}$ fairness in turn implies that

$$\phi_i(v) = \phi_h(v) - \phi_h(w) + \phi_i(w) = F' - \tfrac{C_W}{|W|} + \tfrac{C_V}{|V|}.$$

Since Proposition 3.15(c) also implies that $\phi_j(v) = \phi_i(v)$ for all $i \in T \setminus \{j\}$, it can be concluded that $\phi_j(v) = F' - \frac{C_W}{|W|} + \frac{C_V}{|V|}$. Thus,

$$\phi_i(v) = \begin{cases} F' & \text{for } i \in W \\ F' - \frac{C_W}{|W|} + \frac{C_V}{|V|} & \text{for } i \in V \end{cases} \tag{3.40}$$

Efficiency determines that

$$\sum_{i \in N} \phi_i(v) = n\,F' - \frac{|V|}{|W|} C_W + C_V \equiv C_V + C_W$$

and, thus, $F' = \frac{|W|+|V|}{|V|\,n} C_W = \frac{C_W}{|W|}$. Hence,

$$\phi_i(v) = \begin{cases} \frac{C_W}{|W|} & \text{for } i \in W \\ \frac{C_V}{|V|} & \text{for } i \in V \end{cases} \tag{3.41}$$

2. Suppose that $n = 2$.
   Then $N = \{i, j\}$, $i \neq j$, and $v = C_i\, u_i + C_j\, u_j$. Without loss of generality we may assume that $C_i \geqq C_j$.
   Let $\phi_j(v) = F$. The null-player property implies that $\phi_j\left((C_i - C_j)\, u_i\right) = 0$. With efficiency this in turn implies that $\phi_i\left((C_i - C_j)\, u_i\right) = C_i - C_j$. Fairness now implies that

   $$\phi_i(v) = \phi_j(v) - \phi_j\left((C_i - C_j)\, u_i\right) + \phi_i\left((C_i - C_j)\, u_i\right) = F + C_i - C_j.$$

   Efficiency imposes that

   $$\phi_i(v) + \phi_j(v) = 2F + C_i + C_j \equiv C_i + C_j$$

   yielding that $F = C_j$ and

   $$\begin{aligned} \phi_i(v) &= F + C_i - C_j = C_i \\ \phi_j(v) &= F = C_j \end{aligned}$$

   We may conclude that in this case all values $\phi(v)$ are uniquely determined.

All cases discussed above imply that $\phi(v)$ is uniquely determined by the three properties imposed for all games $v \in \mathcal{G}^N$. Since the Shapley value φ satisfies these three properties, it is clear that $\phi = \varphi$. This proves the assertion.

Proof of Theorem 3.30

We first show that the Myerson value $\mu \colon \mathcal{G}^N \times \mathfrak{W}^N \to \mathbb{R}^N$ indeed satisfies the three stated properties in the assertion. For that purpose let $(v, \Omega) \in \mathcal{G}^N \times \mathfrak{W}^N$ be given.

1. *μ satisfies component-efficiency.*
   If $N \in \Omega$, then $C_\Omega(N) = \{N\}$ and

   $$\sum_{i \in N} \mu_i(v, \Omega) = \sum_{i \in N} \varphi_i(v_\Omega) = v_\Omega(N) = v(N).$$

   Suppose that $N \notin \Omega$ and consider $M \in C_\Omega(N)$. We now define $u^M \in \mathcal{G}^N$ by for every $S \in 2^N$

$$u^M(S) = v_\Omega(S \cap M) = \sum_{T \in C_\Omega(S \cap M)} v(T).$$

Furthermore, we note that $C_\Omega(S) = \cup_{T \in C_\Omega(N)} C_\Omega(S \cap T)$. This in turn implies that $v_\Omega = \sum_{T \in C_\Omega(N)} u^T$. This then leads to

$$\begin{aligned}\sum_{i \in M} \mu_i(v, \Omega) &= \sum_{i \in M} \varphi_i \left(\sum_{T \in C_\Omega(N)} u^T \right) = \\ &= \sum_{i \in M} \varphi_i(u^M) + \sum_{T \in C_\Omega(N):\, T \neq M} \left[\sum_{i \in M} \varphi_i(u^T) \right] = \\ &= v_\Omega(M) + \sum_{T \in C_\Omega(N):\, T \neq M} \left[\sum_{i \in M} 0 \right] = v(M),\end{aligned}$$

in which the last equation follows from Lemma 3.28 implying that $u^T = 0$ and, so, $\varphi_i(u^T) = 0$ for all $T \neq M$.

2. *μ satisfies the isolated player property.*
   For $i \notin \cup C_\Omega(N)$ we have that $C_\Omega(S) = C_\Omega(S \setminus \{i\})$ for all $S \in \Omega$. This implies that $v_\Omega(S) = v_\Omega(S \setminus \{i\})$ and, thus, $\mu_i(v, \Omega) = 0$.
3. *μ satisfies the balanced payoff property.*
   Let $B \in B(\Omega)$, $\Omega' = \overline{B(\Omega) \setminus \{B\}}$, and consider the game $w \in \mathcal{G}^N$ given by $w = v_\Omega - v_{\Omega'}$.
   First, note that $w(S) = 0$ for all $S \not\supseteq B$. Second, for every $j \in B$: $w(S \setminus \{j\}) = 0$, since $B \nsubseteq S \setminus \{j\}$. Hence,

$$\varphi_j(w) = \sum_{S:\, B \subset S} \frac{(s-1)!\,(n-s)!}{n!} w(S)$$

   where $s = |S|$ and $n = |N|$. From this it follows that $\varphi_j(w) = \varphi_k(w)$ for all $k \in B$. Hence, $\mu_j(v_\Omega) - \mu_j(v_{\Omega'}) = \mu_k(v_\Omega) - \mu_k(v_{\Omega'})$. This implies that μ indeed satisfies the balanced payoff property.

This leaves it to be shown that μ is the unique allocation rule on $\mathcal{G}^N \times \mathfrak{W}^N$ that satisfies these three properties.

Let $(v, \Omega) \in \mathcal{G}^N \times \mathfrak{W}^N$ and assume that there are two allocation rules γ^1 and γ^2 that satisfy these three properties. Let $\mathfrak{B}(\Omega) = \{B \in B(\Omega) \mid |B| \geqq 2\}$ be the class of non-unitary supports in Ω. The proof of the assertion is conducted by induction on the number of sets in $\mathfrak{B}(\Omega)$.

Now suppose that there are two allocation rules γ^1 and γ^2 on $\mathcal{G}^N \times \mathfrak{W}^N$ that satisfy these three desired properties. We will show that $\gamma^1 = \gamma^2$.

If $|\mathfrak{B}(\Omega)| = 0$, then $C_\Omega(N) = \{\{i\} \mid \{i\} \in \Omega\}$. Applying component efficiency and the isolated player property we get that $\gamma^1 = \gamma^2$.

Now assume as the induction hypothesis that $\gamma^1(v,\Omega') = \gamma^2(v,\Omega')$ for all Ω' with $|\mathfrak{B}(\Omega')| \leqq k-1$ and let $|\mathfrak{B}(\Omega)| = k$.

Consider $C \in \mathfrak{B}(\Omega)$. Balanced payoffs now implies that there exist numbers $c, d \in \mathbb{R}$ such that for all $j \in C$:

$$\gamma_j^1(v,\Omega) - \gamma_j^1(v,\overline{B(\Omega)\setminus\{C\}}) = c, \tag{3.42}$$

$$\gamma_j^2(v,\Omega) - \gamma_j^2(v,\overline{B(\Omega)\setminus\{C\}}) = d. \tag{3.43}$$

Now by the induction hypothesis for $j \in C$:

$$\gamma_j^1(v,\overline{B(\Omega)\setminus\{C\}}) = \gamma_j^2(v,\overline{B(\Omega)\setminus\{C\}})$$

So there is a constant $a = c - d$ such that

$$\gamma_j^1(v,\Omega) - \gamma_j^2(v,\Omega) = a \ \text{ for all } j \in C. \tag{3.44}$$

Given $M \in C_\Omega(N)$, by component efficiency of γ^1 and γ^2 we arrive at

$$\sum_{i\in M} \left[\gamma_j^1(v,\Omega) - \gamma_j^2(v,\Omega) \right] = 0$$

On the other hand, it is easy to see that for every $i,j \in M$ there exists a non-unitary supports sequence $B_1, B_2, \ldots, B_p$ in $\mathfrak{B}(\Omega)$, contained in M such that $i \in B_1$, $j \in B_p$ and $B_q \cap B_{q+1} \neq \varnothing$ for $q = 1,\ldots,p-1$.

Using equality (3.44) we have $\gamma_k^1(v,\Omega) - \gamma_k^2(v,\Omega) = a$ for $k \in B_1$. As $B_1 \cap B_2 \neq \varnothing$, there exists $h \in B_1 \cap B_2$ such that $\gamma_h^1(v,\Omega) - \gamma_h^2(v,\Omega) = a$. Thus, applying the property above of the constructed sequence as well as Equation (3.44) recursively for all elements in the sequence $B_1, B_2, \ldots, B_p$ we get that

$$\gamma_i^1(v,\Omega) - \gamma_i^2(v,\Omega) = \gamma_j^1(v,\Omega) - \gamma_j^2(v,\Omega).$$

Hence, $\gamma_i^1(v,\Omega) - \gamma_i^2(v,\Omega) = a$ for all $i \in M$, $M \in C_\Omega(N)$. This in turn implies that

$$\sum_{i\in M} \left[\gamma_j^1(v,\Omega) - \gamma_j^2(v,\Omega) \right] = |M|\, a.$$

Therefore, $|M|\, a = 0$ and hence $\gamma_j^1(v,\Omega) = \gamma_j^2(v,\Omega)$ for all $j \in M$.

This completes the proof of Theorem 3.30.

3.6 Problems

Problem 3.1 Provide a detailed proof of the probabilistic formulation (3.4) by deriving it from the standard formulation (3.3).

Problem 3.2 Consider the value ψ^1 introduced in Example 3.8. Show that this value indeed satisfies the null-player property, symmetry and additivity. Use a counter-example to show that this value indeed is not efficient as claimed.

Problem 3.3 Provide a proof of the following properties:

(a) Show that if a value ϕ on $\mathcal{G}^N$ satisfies Young's strong monotonicity property, then it also has to satisfy Shubik's coalitional monotonicity property.
(b) Show that the Shapley value φ satisfies strong monotonicity.
(c) Consider the monotonicity property introduced by van den Brink (2007). A value ϕ is *B-monotone* if for every player $i \in N$ and every pair of games $v, w \in \mathcal{G}^N$ with $v(S) \geqq w(S)$ for all $S \subset N$ with $i \in S$ it holds that $\phi_i(v) \geqq \phi_i(w)$.

   (i) Show that B-monotonicity implies Shubik's coalitional monotonicity.
   (ii) Also construct two counter examples that show that B-monotonicity and strong monotonicity do not imply one another.

Problem 3.4 (*Nullifying player properties*) Recall that the *egalitarian value* is given by $\psi^e \colon \mathcal{G}^N \to \mathbb{R}^N$ with

$$\psi_i^e(v) = \frac{v(N)}{n} \quad \text{for all } i \in N. \tag{3.45}$$

A variation of the egalitarian solution is the *equal surplus division rule* given by $\psi^{es} \colon \mathcal{G}^N \to \mathbb{R}^N$ with

$$\psi_i^{es}(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n} \quad \text{for all } i \in N. \tag{3.46}$$

Now, van den Brink (2007) introduces a property concerning the notion of a "nullifying" player. These players reduce the value of a coalition to zero. More precisely, a player $i \in N$ is a *nullifying player* in the game v if $v(S) = 0$ for every $S \subset N$ with $i \in S$.

Now a value ϕ satisfies the *nullifying player property* if $\phi_i(v) = 0$ for every nullifying player $i \in N$ in the cooperative game $v \in \mathcal{G}^N$.

(a) As van den Brink (2007) first showed, if one replaces the null player property in Shapley's seminal axiomatization of his value, one arrives at an axiomatization of the egalitarian solution. Now prove that a value ϕ is equal to the egalitarian solution ψ^e if and only if ϕ satisfies efficiency, symmetry, additivity and the nullifying player property.
(b) We can simplify this axiomatization further by replacing additivity and the nullifying player property by a single axiom: A value ϕ satisfies *null additivity* if for every pair of games $v, w \in \mathcal{G}^N$ and every player $i \in N$ it holds that $\phi_i(v + w) = \phi_i(v)$ if i is a nullifying player in w.

Show that the value ϕ is equal to the egalitarian solution ψ^e if and only if ϕ satisfies efficiency, symmetry and null-additivity.

(c) Next consider the additional property of invariance. A value ϕ is *invariant* if for every game $v \in \mathcal{G}^N$, real number $\alpha \in \mathbb{R}$ and vector $\beta \in \mathbb{R}^N$ it holds that $\phi(\alpha v + \beta) = \alpha\,\phi(v) + \beta$, where $\alpha v + \beta \in \mathcal{G}^N$ is defined by $(\alpha v + \beta)(S) = \alpha v(S) + \sum_{j \in S} \beta_j$ for every $S \subset N$.

Prove that a value ϕ is equal to the equal surplus division rule ψ^{es} if and only if ϕ satisfies efficiency, symmetry, additivity, the nullifying player property for the class of zero-normalized games, as well as invariance.

Problem 3.5 Let N be some player set. Show that the union stable cover of some coalitional structure $\Omega \subset 2^N$ is equal to the union stable cover of its basis $B(\Omega)$.

Problem 3.6 Let Ω be some coalitional structure and let $S \in C_\Omega(N)$ be some Ω-component of the player set N. Show that for every $i,j \in S$ with $i \neq j$ there exists some sequence of non-unitary supports $B_1, B_2, \ldots, B_p$ in $\mathfrak{B}(\Omega)$, contained in S such that $i \in B_1, j \in B_p$ and $B_q \cap B_{q+1} \neq \varnothing$ for $q = 1, \ldots, p-1$.

Problem 3.7 Consider a coalition structure in the sense of Aumann and Drèze (1974) denoted by $\Omega_a = \{\varnothing, N_1, \ldots, N_m\}$ on player set N. Here, for every $k = 1, \ldots, m$ we assume that $N_k \neq \varnothing$ and that Ω_a forms a partitioning of N.

(a) Determine the support basis $B(\Omega_a)$ of the given coalition structure Ω_a. Furthermore, for this particular case determine the union stable cover $\overline{\Omega_a}$, the global components $C_{\Omega_a}(N)$, and the Ω_a-restriction of any arbitrary game v.
(b) Check Myerson's Characterization Theorem 3.30 for the given coalitional structure Ω_a.

Problem 3.8 We recall from the discussion in the previous chapter that a communication network on N is a set of communication links $g \subset \{ij \mid i,j \in N\}$, where $ij = \{i,j\}$ is a binary set representing a communication link between players i and j. Hence, if $ij \in g$, then it is assumed that players i and j are able to communicate with each other.

Two players i and j are now connected in the network g if these two players are connected by a path in the network, i.e., there exist $i_1, \ldots, j_K \in N$ such that $i_1 = i$, $i_K = j$ and $i_k i_{k+1} \in g$ for all $k = 1, \ldots, K-1$. Now a group of players $S \subset N$ is connected in the network g if all members $i,j \in S$ are connected in g. Now Myerson (1977) introduced

$$\Omega_g = \{S \subset N \mid S \text{ is connected in } g\}.$$

The coalitional structure Ω_g is the class of connected coalitions in the communication network g. A *network situation* is now given by (v, g), where $v \in \mathcal{G}^N$ and g is a communication network on N.

The Myerson value for communication situations is now defined by $\mu(v, g) = \mu(v, \Omega_g)$.

(a) Show that for this given communication situation, the class of supports of Ω_g is given by

$$B(\Omega_g) = \{\varnothing\} \cup \{\,\{i\} \mid i \in N\} \cup g. \tag{3.47}$$

(b) Consider some allocation rule ϕ on the class of communication situations. Restate component-efficiency, the isolated player property as well as the balanced payoff property for such communication situations.
(c) Show that the Myerson value μ defined above is the unique allocation rule on the class of communication situations that satisfies these three re-stated properties.[15]

Problem 3.9 (*The Proper Shapley Value*) Vorob'ev and Liapounov (1998) introduced the so-called "proper" Shapley value. This value is the fixed point of a Shapley correspondence. In order to define this construction, let N be some player set and let $w \in \mathbb{R}^N_+$ be an N-dimensional weight vector. Now the weight of player i is given by $w_i \geqq 0$ and the weight of coalition $S \subset N$ is computed as $w(S) = \sum_{i \in S} w_i$. The w-weighted Shapley value of the game $v \in \mathcal{G}^N$ is now for every player $i \in N$ defined by

$$\varphi_i^w(v) = \sum_{S \subset N:\, i \in S} \frac{w_i}{w(S)} \cdot \Delta_v(S). \tag{3.48}$$

We can obtain the (regular) Shapley value by taking equal weights. Of course, the weight vector w can be normalized to satisfy $\sum_{i \in N} w_i = 1$, i.e., w can be selected from the $(|N| - 1)$-dimensional unit simplex

$$\mathcal{S}^{|N|-1} = \left\{ w \in R^N_+ \;\middle|\; \sum_{i \in N} w_i = 1 \right\}$$

Let $v \in \mathcal{G}^N$ be a given game. Then the *Shapley mapping* $\Phi^v : \mathcal{S}^{|N|-1} \to \mathbb{R}^N$ assigns to every weight vector w in the corresponding simplex the Shapley value $\Phi^v(w) = \varphi^w(v)$.

(a) Show that for every game $v \in \mathcal{G}^N$ with $v(N) = 1$ and $\Delta_v(S) \geqq 0$ for every coalition $S \subset N$, there exists a unique fixed point of the Shapley mapping Φ^v, i.e., there exists a unique weight vector $w^* \in \mathbb{R}^N : \Phi^v(w^*) = w^*$.
This fixed point is denoted as the *proper Shapley value* of the game v.
(b) Show that the proper Shapley value w^* for the game $v \in \mathcal{G}^N$ with $v(N) = 1$ and $\Delta_v(S) \geqq 0$ for every coalition $S \subset N$, is exactly the solution to

[15] This is the original characterization of the Myerson value developed in Myerson (1977).

$$\max_{x \in \mathcal{S}^{|N|-1}} \prod_{S \subset N} \left(\sum_{j \in S} x_j \right)^{\Delta_v(S)} \tag{3.49}$$

It should be clear that this connects the proper Shapley value to the Nash bargaining solution.

Problem 3.10 (*The Kikuta-Milnor Value*) An alternative to the Shapley value is the family of so-called "compromise" values. These values are based on a reasonable upper and lower bound for the payoffs to the players in the game. The value that is generated by these upper and lower bounds is exactly the convex combination of these bounds that is efficient. Here I consider the Kikuta-Milnor compromise value based on a minimal and maximal marginal value for all players as these respective bounds.

Let $v \in \mathcal{G}^N$ and define for every $i \in N$ her upper bound as the maximal marginal contribution given by

$$\mu_i^u(v) = \max_{S \subset N:\, i \in S} v(S) - v(S - i) \tag{3.50}$$

and her lower bound as the minimal marginal contribution given by

$$\mu_i^l(v) = \min_{S \subset N:\, i \in S} v(S) - v(S - i) \tag{3.51}$$

The Kikuta-Milnor value $\kappa : \mathcal{G}^N \to \mathbb{R}^N$ is now defined by

$$\kappa(v) = \lambda\, \mu^u(v) + (1 - \lambda)\, \mu^l(v) \tag{3.52}$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} \kappa_i(v) = v(N)$. Prove the following statements with regard to this compromise value.

(a) For every $v \in \mathcal{G}^N$: $\mu^u(v) \geqq \mu^l(v)$.
(b) The Kikuta-Milnor index is well defined on the space of all games $\mathcal{G}^N$ on player set N. Provide a well-defined expression of the Kikuta-Milnor value function κ.
(c) Under which conditions on v does it hold that $\lambda \in [0, 1]$? Be complete in your answer.
(d) If v is superadditive, then $\kappa(v) \in I(v)$. Is the reverse also true? If it is, then provide a proof. Otherwise, construct a counter example.
(e) If $v \in \mathcal{G}^N$ with $n = 3$ is superadditive, then

$$\kappa(v) \in C(v) \text{ implies } \varphi(v) \in C(v).$$

(f) Recall the notion of the dual game of $v \in \mathcal{G}^N$ as the game $v^\star \in \mathcal{G}^N$ defined by $v^\star(S) = v(N) - v(N \setminus S)$, $S \subset N$. It holds that $\kappa(v) = \kappa(v^\star)$.

(g) Consider a convex game $v \in \mathcal{G}^N$. What can be said about the relationship of the Kukita-Milnor value $\kappa(v)$ and the Core $C(v)$? Explain your conclusions in detail.

Problem 3.11 (*The CIS value*) Consider another set of upper and lower bounds for a game to define another compromise value. For the game $v \in \mathcal{G}^N$ I select this time for every player $i \in N$ the following obvious lower and upper bounds:

$$m_i^l(v) = v(i) \tag{3.53}$$

and

$$m_i^u(v) = v(N) - \sum_{i \in N} m_i^l(v) = v(N) - \sum_{i \in N} v(i) \tag{3.54}$$

The CIS compromise value is now defined by

$$\gamma(v) = \lambda\, m^u(v) + (1 - \lambda)\, m^l(v) \tag{3.55}$$

where $\lambda \in [0, 1]$ is such that $\sum_{i \in N} \gamma_i(v) = v(N)$. Prove the following statements with regard to the CIS compromise value.

(a) The CIS value is well-defined for the class of weakly essential games, i.e., those games $v \in \mathcal{G}^N$ such that $\sum_N v(i) \leqq v(N)$. Compute an exact expression for the CIS compromise value γ on the class of all weakly essential games.
(b) If $v \in \mathcal{G}^N$ with $n = 3$ is superadditive, then

$$\gamma(v) \in C(v) \text{ implies } \varphi(v) \in C(v).$$

(c) Compute $\gamma(v^\star)$ for any weakly essential game v, where $v^\star$ is the dual of v.

Chapter 4
The Cooperative Potential

In the previous chapters I investigated different solution concepts for cooperative games as descriptions of productive interactive situations. First, I considered the Core, which is a set of payoff vectors based on the bargaining power of various coalitions in the interactive situation. Next I considered single valued solution concepts with very powerful properties, resulting into value theory. The strength of values as solution concepts is that they assign a *unique* allocation or imputation to every cooperative game. In this regard, values reduce the information contained in a characteristic function into a single vector.

In this chapter I consider the next step into this reductionist process: Hart and Mas-Colell (1989) introduced a function that assigns to every possible cooperative game a unique *number*. This is the so-called *potential value* of the cooperative game in question. Payoffs—or allocations—can then be determined through the standard rule of allocating each player's marginal contribution to the potential of a game; i.e., each player receives exactly his or her marginal contribution to the potential of that particular cooperative game. The resulting payoff vector is efficient and unique if the potential function satisfies a simple additivity property. Furthermore, Hart and Mas-Colell (1989) show that this payoff vector is exactly the Shapley value of the cooperative game under consideration.

We can summarize the main conclusion from this theory by quoting the main seminal papers on this subject:

> There exists a unique real function on [the space of all cooperative] games, called the *potential function*, with respect to which the marginal contributions of all players are always efficient. Moreover, these marginal contributions are precisely the Shapley (1953) value. (Hart and Mas-Colell, 1988, p. 128)

In more mathematical terms the main insight from this theory is that there is a unique potential function that is efficient and its (discrete) gradient is exactly the Shapley value. This has some further consequences in the sense that this insight can be used to develop an alternative axiomatization of the Shapley value based on a consistency axiom.

The concept of a potential function has been used extensively in physics. The notion of a "potential" has been introduced seminally by Daniel Bernoulli in 1738. One of the most powerful applications of the idea of a potential is the theory of

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gravitational fields. It describes exactly conditions under which there is path independence; the energy used to travel along a certain path is always the same in such gravitational fields. This property has been exploited by Hart and Mas-Colell (1989) in their analysis of the Shapley value based on their potential function.

The first part of tis chapter is largely based on Hart and Mas-Colell (1989). This paper can be denoted as the seminal work on potential theory. This contribution is so exhaustive that it did not leave many significant questions to be resolved.

Second, I discuss several extensions of the foundations laid by Hart and Mas-Colell. The extension developed by Calvo and Santos (1997) in some sense reverses the main insight from Hart and Mas-Colell (1989) that the marginals of the cooperative potential are exactly the Shapley values of the players in a cooperative game. Instead, Calvo and Santos consider for any allocation rule or value function the construction of a potential function that has this property. Thus, they broaden the class of allocation rules that are related to such value-based potentials.

The concluding part of this chapter is devoted to the discussion of another extension of the cooperative potential concerning share values. Share potentials are seminally developed by van den Brink and van der Laan (2007), which in turn is based on the work by Calvo and Santos (1997). This approach identifies a class of cooperative potential functions corresponding to a class of so-called *share functions*, which generalize the Shapley value. The main result here is, however, that the share functions in this class can be derived from the Shapley share function.

4.1 A Potential Function for Cooperative Games

In this chapter I consider the space of *all* cooperative games with transferable utilities. Hence, I no longer require our analysis to be restricted to a given player set. Instead I consider the space of cooperative games being the union of all sets of cooperative games on finite player sets.

I denote by $\mathcal{G}$ the space of all cooperative games defined on finite player sets:

$$\mathcal{G} = \bigcup_{N:\ \#N \in \mathbb{N}} \left\{ (N, v) \;\middle|\; v \in \mathcal{G}^N \right\} \tag{4.1}$$

In the sequel we denote by $(N, v) \in \mathcal{G}$ a (finite) player set N and a cooperative game $v \in \mathcal{G}^N$ that is defined on it. This construction allows us to compare cooperative games on different player sets. For example, let $(N, v) \in \mathcal{G}$ and $M \subset N$ with $M \neq N$. Then with some abuse of notation we denote by $(M, v) \in \mathcal{G}$ the *restriction* of v to M given by $v(S) = v(S \cap M)$ for all $S \subset M$.

Let $P\colon \mathcal{G} \to \mathbb{R}$ be some real valued function on the space of all cooperative games. The function associates with every game $(N, v) \in \mathcal{G}$ some real number $P(N, v) \in \mathbb{R}$. We define the *marginal contribution* of player $i \in N$ to game (N, v) by

$$D_i P(N, v) = P(N, v) - P(N - i, v) \tag{4.2}$$

where $N - i = N \setminus \{i\}$ and $(N - i, v)$ is the restriction of (N, v) to $N - i$.

Definition 4.1 (Hart and Mas-Colell, 1989) A function $\Psi\colon \mathcal{G} \to \mathbb{R}$ is a *HM-potential function* on $\mathcal{G}$ if it holds that for every game $(N, v) \in \mathcal{G}$:

$$\sum_{i \in N} D_i \Psi(N, v) = v(N) \tag{4.3}$$

and $\Psi(\varnothing, v) = 0$ for every $(\varnothing, v) \in \mathcal{G}$.

In the remainder of this chapter we denote a HM-potential function simply as the *cooperative potential*. In the last section of this chapter we introduce a class of related functions—denoted as μ-potential functions—but the followed terminology will not lead to any confusion.

The definition of a potential function on the space of all cooperative games requires that the function is efficient in the sense that the marginal contributions of the players in each game add up exactly to the total value generated in that game. The next result states the main insight from Hart and Mas-Colell (1989). For a proof of this result we refer to the appendix of this chapter.

Theorem 4.2 (Hart and Mas-Colell, 1989, Theorem A)

(a) *There exists a unique potential function Ψ on $\mathcal{G}$.*
(b) *The unique potential function Ψ on the space of cooperative games $\mathcal{G}$ is given by the following equivalent formulations:*

$$\Psi(N, v) = \sum_{S \subset N} \frac{(|S| - 1)!\,(n - |S|)!}{n!} v(S) \tag{4.4}$$

$$\Psi(N, v) = \sum_{S \subset N} \frac{\Delta_v(S)}{|S|} \tag{4.5}$$

$$\Psi(N, v) = \int_0^1 \frac{E_v(t, \dots, t)}{t}\, dt \tag{4.6}$$

$$\Psi(N, v) = |N| \cdot \mathbb{E}_S \left[\frac{v(S)}{|S|} \right] \tag{4.7}$$

where in the last formulation $\mathbb{E}_S$ stands for taking the expectation with regard to the formation of coalitions according to standard Weber strings such that these Weber strings form with equal probability.

The potential function on the space of all cooperative games $\mathcal{G}$ can be computed in various different ways as stated in Theorem 4.2(b). Each formulation has its own merits.

The first formulation is a straightforward formula for the potential based on the actual values generated by the various coalitions in the game. It is a formulation that is followed throughout the literature, in particular since the weights of the various coalitions in this formulation are the Shapley weights used in the original formulation of his value by Shapley (1953).

The second formulation states that the potential of a game can also be computed as the total sum of all equally distributed Harsanyi dividends over the members of the various value generating coalitions. This formulation allows us to compute the potential of a game directly from the generated Harsanyi dividends.

The two remaining formulations provide a more detailed insight into the nature of the potential function. The third formulation namely links the potential to the multilinear extension of a game. Using the probabilistic interpretation of the MLE, the potential is the integral of the MLE where all players have equal probability of being member of a coalition.

The fourth and final formulation shows that the potential of a game is the expectation of the total value allocated if all players receive an equal share of the coalitional values generated. The only caveat here is that coalitions are formed through Weber strings, i.e., players enter the game in random order and coalitions are formed in order of entry.

The second formulation based on the Harsanyi dividends of a game also leads to a full characterization of the marginal contributions as the Shapley value of that game.

Corollary 4.3 *For every game $(N, v) \in \mathcal{G}$ the marginal contributions of the players are equal to their respective Shapley values, i.e., for every $i \in N$:*

$$D_i \Psi(N, v) = \varphi_i(N, v). \tag{4.8}$$

Proof Consider the second formulation given in Theorem 4.2(b). It now follows that

$$D_i \Psi(N, v) = \sum_{S \subset N} \frac{\Delta_v(S)}{|S|} - \sum_{T \subset N - i} \frac{\Delta_v(T)}{|T|} = \sum_{S:\, i \in S} \frac{\Delta_v(S)}{|S|} = \varphi_i(N, v) \tag{4.9}$$

This completes the proof of the corollary. ■

To illustrate these results I introduce the following simple example. This example is a slight modification of the trade example that I developed in Examples 1.2, 1.5, and 1.9.

Example 4.4 Consider a trade situation of one seller and two buyers, similar to Example 1.2. The reservation values for the seller is 10, while the two buyers are willing to pay up to 130 and 70, respectively. This trade situation can now be represented by a game in characteristic function form. (In this game we take the seller's value of 10 as a basis value. Thus, we are not only considering the division of the net gains from trade, rather the gross gains from trade.)

Let $N = \{1, 2, 3\}$, where player 1 is the seller, player 2 is buyer #1 and player 3 is buyer #2, and consider the game $v \in \mathcal{G}^N$ given by the following table:

Coalition S	$v(S)$	$\Delta_v(S)$	$\Psi(S,v)$
1	10	10	10
2	0	0	0
3	0	0	0
12	130	120	70
13	70	60	40
23	0	0	0
123	130	–60	80

The potential values for each coalition in this game are computed according to the original definition. For example, $\Psi(12,v) = \Delta_v(1) + \Delta_v(2) + \frac{\Delta_v(12)}{2} = 10 + \frac{120}{2} = 70$.

From the values reported in the table we deduce that

$$D_1\Psi(N,v) = \Psi(N,v) - \Psi(23,v) = 80 - 0 = 80$$
$$D_2\Psi(N,v) = \Psi(N,v) - \Psi(13,v) = 80 - 40 = 40$$
$$D_3\Psi(N,v) = \Psi(N,v) - \Psi(12,v) = 80 - 70 = 10$$

These values indeed coincide with the Shapley value φ of the game v:

$$\varphi_1(N,v) = \frac{10}{1} + \frac{120}{2} + \frac{60}{2} - \frac{60}{3} = 80$$
$$\varphi_2(N,v) = \frac{120}{2} - \frac{60}{3} = 40$$
$$\varphi_1(N,v) = \frac{60}{2} - \frac{60}{3} = 10$$

It can easily be concluded that the introduction of the potential function Ψ simplifies the calculation of all Shapley values for the various games. In fact this can be represented by extending the previous table to include all Shapley values for the sub-games of (N,v):

Coalition S	v(S)	$\Psi(S,v)$	$\varphi_1(S,v)$	$\varphi_2(S,v)$	$\varphi_3(S,v)$
1	10	10	10	–	–
2	0	0	–	0	–
3	0	0	–	–	0
12	130	70	70	60	–
13	70	40	40	–	30
23	0	0	–	0	0
123	130	80	80	40	10

As an aside I note that the calculations of all Shapley values reveal that the seller gets much more leverage in the bargaining over the distribution of the gains from trade when two buyers are present rather than only one. ■

4.2 The Cooperative Potential and the Shapley Value

Hart and Mas-Colell (1989) also developed an alternative approach to analyze the relationship of the cooperative potential Ψ and the Shapley value φ.

We first introduce some terminology. A mapping ϕ on $\mathcal{G}$ is called an *allocation rule* on $\mathcal{G}$ if for every $(N, v) \in \mathcal{G}$ it holds that $\phi(N, v) \in \mathbb{R}^N$. The *Shapley value* on $\mathcal{G}$ is the allocation rule φ on $\mathcal{G}$ given by $\varphi(N, v) = \varphi(v) \in \mathbb{R}^N$ for every $(N, v) \in \mathcal{G}$, recalling that in that case $v \in \mathcal{G}^N$.

With the notion of an allocation rule on the class of all cooperative games G, we can properly address the formulation of fairness or balanced payoff conditions for this most general class of cooperative games.

This approach is relational and based on side payment differences between pairs of players. Given a description of the differences between the payoffs for various players, we can look at allocation rules that implement these differences. The most natural type of differences occur if players are treated fairly with regard to the marginal differences of their respective payoff. This difference system leads exactly to the Shapley value.

Formally, a *difference system* is a function $d : N \times N \to \mathbb{R}$ that assigns to every pair of players $i, j \in N$ a difference $d_{ij} \in \mathbb{R}$.

Definition 4.5 A difference system $d : N \times N \to \mathbb{R}$ is *compatible* if the following three properties hold:

(i) $d_{ii} = 0$ for all $i \in N$,
(ii) $d_{ij} = -d_{ji}$ for all $i, j \in N$, and
(iii) $d_{ij} + d_{jh} = d_{ih}$ for all $i, j, h \in N$.

An allocation $x \in \mathbb{R}^N$ *preserves the difference system d* if for all players $i, j \in N$ it holds that $x_i - x_j = d_{ij}$.

A compatible difference system is one in which the differences are systematically distributed. The properties that a compatible difference system satisfies exactly describe that difference add up to zero over all closed loops in the player set. So, from moving from one player to the next and returning in that fashion to the original player leads to a total difference of exactly zero.

We recall that an allocation x is efficient if $\sum_N x_i = v(N)$. The next result shows that there is exactly one allocation that is efficient and preserves the difference imposed by a compatible difference system.

Proposition 4.6 *For every compatible difference system d, there exists a unique efficient allocation x that preserves the difference system d. This efficient allocation x is given by*

$$x_i = \frac{1}{n} \left[v(N) + \sum_{j \neq i} d_{ij} \right] \tag{4.10}$$

Proof Let $i, j \in N$ with $i \neq j$. Then we see that $x_i - x_j = d_{ij}$. Indeed,

$$\begin{aligned} n \cdot x_i - n \cdot x_j &= \left[v(N) + \sum_{h \neq i} d_{ih} \right] - \left[v(N) + \sum_{h \neq j} d_{jh} \right] = \\ &= \sum_{h \neq i,j} \left(d_{ih} - d_{jh} \right) + d_{ij} - d_{ji} = \\ &= \sum_{h \neq i,j} \left(d_{ih} + d_{hj} \right) + d_{ij} + d_{ij} = \\ &= \sum_{h \neq i,j} \left(d_{ij} \right) + 2\, d_{ij} = n \cdot d_{ij} \end{aligned}$$

Hence, x indeed preserves the differences imposed by difference system d.

Next we show efficiency. Indeed, given the preservation of difference we get

$$\sum_N x_i = x_1 + \sum_{j \neq 1} \left(x_1 - d_{1j} \right) = n \cdot x_1 - \sum_{j \neq 1} d_{1j}.$$

With the definition of x_1 given we thus note that

$$\sum_N x_i = \left[v(N) + \sum_{j \neq 1} d_{1j} \right] - \sum_{j \neq 1} d_{1j} = v(N).$$

This shows the assertion. ■

We conclude that every compatible difference system d, thus, induces an efficient allocation that preserves these differences. It is now our aim to use the introduced notion of a difference system to "reconstruct" the Shapley value.

Let ϕ be some allocation rule on the class of all cooperative games $\mathcal{G}$. Let $\phi(N, v)$ be given. Now for every $S \subsetneq N$, we define $\phi(S, v)$ as the allocation assigned to the subgame (S, v) of (N, v). Then we can introduce the difference system $d^{\phi} : N \times N \to \mathbb{R}$ by

$$d^{\phi}_{ij} = \phi_i(N - j, v) - \phi_j(N - i, v). \tag{4.11}$$

This is exactly the difference in payoff between these two players if player i would operate without player j and vice versa. According to Hart and Mas-Colell this difference system exactly imposes a natural way to compare the "relative positions" of the various players in the game under consideration based on the allocation rule ϕ.

The introduction of the difference system d^{ϕ} above leads to a recursive definition of an allocation rule that assigns exactly the allocations that preserve the differences imposed by d^{ϕ}. For its proper introduction we need to define the initiating values for that allocation rule. The main result is that there exists a unique efficient allocation rule that preserves these natural differences using the most natural starting allocation and that it is exactly the Shapley value.

For a proof of the next assertion I refer to the appendix of this chapter.

Theorem 4.7 *The Shapley value φ is the unique efficient allocation rule ϕ on $\mathcal{G}$ such that for every player $i \in N$:*

$$\phi_i(\{i\}, v) = v(\{i\}), \tag{4.12}$$

and such that ϕ preserves the difference system d^{ϕ}.

Theorem 4.7 states that the Shapley value is essentially self-referential, or *consistent*. Indeed, the Shapley value preserves its own difference system. Consistency will be developed in full detail in the next section of this chapter.

To provide some insights from the proof of Theorem 4.7, I remark that from the given starting condition (4.12) and formula (4.10) given in Proposition 4.6 we derive that for a two-player situation it has to hold that

$$\begin{aligned} x_i(ij, v) &= \frac{1}{2}\left[\, v(ij) + \delta_{ij} \,\right] = \frac{1}{2}\left[\, v(ij) + x_i(i) - x_j(j) \,\right] = \\ &= \frac{1}{2}\left[\, v(ij) + v(i) - v(j) \,\right] = v(i) + \frac{1}{2}\left[\, v(ij) - v(i) - v(j) \,\right] = \\ &= v(i) + \frac{1}{2}\,\Delta_v(ij). \end{aligned}$$

Hence, the introduction of an efficient allocation ϕ that preserves the natural difference system d^{ϕ} for a two-player game is exactly the fair allocation of the natural surplus.

From Theorem 4.7 and the definition of the "natural" difference system d^{ϕ} we can now conclude that the Shapley value satisfies the further appealing property that Myerson (1977) introduced for network-restricted cooperative games. Here this translates to a very natural extension on the class of all cooperative games $\mathcal{G}$:

Definition 4.8 An allocation rule ϕ on G satisfies (Myerson's) *balanced payoff property* if for all pairs of players $i, j \in N$ it holds that

$$\phi_i(N, v) - \phi_i(N - j, v) = \phi_j(N, v) - \phi_j(N - i, v) \tag{4.13}$$

As discussed in Chapter 3, this balanced payoff property was first introduced by Myerson (1980) for the class of all cooperative games $\mathcal{G}$. (A related property for games with coalitional structures we discussed in the section on the Myerson value in Chapter 3.) Here, the balanced payoff property refers to the equality of the change in payoff if different players join a coalition.

The balanced payoff property provides a direct link with the Myerson value for arbitrary coalitional structures, discussed in the previous chapter. Namely, the Myerson value was fully characterized by this property and component efficiency.

It turns out that the Shapley value—being the Myerson value without constraints on coalition formation—is uniquely determined by the balanced payoff property and efficiency. This characterization of the Shapley value is an easy consequence of Theorem 4.7 as well as Theorem 3.30:

Corollary 4.9 (Myerson, 1980) *There exists a unique efficient allocation rule that satisfies the balanced payoff property and it is equal to the Shapley value* φ.

I discuss a simple example to illustrate the concept of balanced payoffs and the use of the cooperative potential to characterize the Shapley value.

Example 4.10 Consider $N = \{1, 2, 3\}$ and a game (N, v) given in the following table. This table includes the Harsanyi dividends as well as the Shapley values for all players.

S	$v(S)$	$\Delta_v(S)$	$\varphi_1(S, v)$	$\varphi_2(S, v)$	$\varphi_3(S, v)$
1	0	0	0	–	–
2	2	2	–	2	–
3	4	4	–	–	4
12	5	3	$1\frac{1}{2}$	$3\frac{1}{2}$	–
13	5	1	$\frac{1}{2}$	–	$4\frac{1}{2}$
23	8	2	–	3	5
123	12	0	2	$4\frac{1}{2}$	$5\frac{1}{2}$

Next consider the balanced payoff property for the computed Shapley values φ in this example. Consider the balance of players 1 and 2:

$$\varphi_1(N, v) - \varphi_1(13, v) = 2 - \tfrac{1}{2} = 1\tfrac{1}{2}$$
$$\varphi_2(N, v) - \varphi_2(23, v) = 4\tfrac{1}{2} - 3 = 1\tfrac{1}{2}$$

This indeed illustrates the balanced payoff property of the Shapley value. ■

4.3 Consistency and the Reductionist Approach

The discussion in the previous section on the foundation of the Shapley value on difference systems and an initial definition for two-player games can be explored further. The basic idea is to define a value on a sensible allocation for a two-player game and then to recursively build it up through consistently applying a definition of the value for smaller games to determine the value for the larger game. Hence, knowing the value for two-player games, one can determine the value for a three player game by decomposing the game according to some reduction rule into two-player games and apply the defined value for those.

The approach is based on what is called *consistency* and the proper definition of what constitutes a *reduced game*. A value that can be constructed using consistency and some form of a reduced game is also called a *consistent* value. This approach in general can be indicated as the reductionist approach to value theory.

Consistency itself is a basic property that always applies in the same fashion. However, a reduced game can be constructed in multiple fashions. The first authors to seminally develop this reductionist approach were Davis and Maschler (1965). Very interesting follow-ups based on this reductionist approach were developed by Sobolev (1975), Aumann and Maschler (1985) and Moulin (1985). I will return to a discussion of these contributions at the end of this section.

In this section I first develop the reductionist approach to the Shapley value seminally formulated in Hart and Mas-Colell (1989). Subsequently I shortly discuss several alternative formulations of a reduced game, introduced by Davis and Maschler (1965) and Moulin (1985), respectively. The Davis-Maschler reduced game leads to a reductionist characterization of the *pre-nucleolus*. The reduced game definition of Moulin results into a reductionist characterization of Moulin's *equal allocation with non-separable costs*.

Throughout the remainder of our discussion I let ϕ be some value function on the universe of all cooperative games $\mathcal{G}$. As usual we write $\phi(N, v) \in \mathbb{R}^N$ for the solution itself and $\phi_i(N, v) \in \mathbb{R}^N$ for the allocation to player $i \in N$ under the value function ϕ. We now introduce the reductionist approach developed by Hart and Mas-Colell (1989).

Definition 4.11 (The HM-reduction) Let (N, v) and $T \subset N$ be given. The *HM-reduced game* (T, v_T^ϕ) on T is now for every coalition $S \subset T$ defined by

$$v_T^\phi(S) = v(S \cup T^c) - \sum_{i \in T^c} \phi_i(S \cup T^c, v) \tag{4.14}$$

where $T^c = N \setminus T$ is the complement of coalition T.

The value function ϕ is now *HM-consistent* if for every game (N, v) and every coalition $T \subset N$:

$$\phi_i(N, v) = \phi_i(T, v_T^\phi) \tag{4.15}$$

for every player $i \in T$.

The HM-reduced game describes a reasonable value that can generated by a coalition $S \subset T$ given that the value function ϕ is the accepted norm of how to allocate a payoff to the players outside the prevailing coalition T. In other words, the HM-reduced game assigns to every coalition S the wealth that coalition S can generate together with players *outside* T given that those players are paid according to the value function ϕ.[1]

To elaborate on this interpretation I consider the case of an efficient value function ϕ on $\mathcal{G}$. In that case (4.14) can be rewritten as

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[1] I note here that Naumova (2005) has extended the discussion of HM-consistency to nonsymmetric values. I refer to that paper for an elaborate analysis of the HM-consistency property o characterize a broad class of allocation rules other than the Shapley value.

$$v_T^{\phi}(S) = \sum_{i \in S} \phi_i(S \cup T^c, v)$$

for any $S \subset T \subset N$. Furthermore, it follows from the above that

$$v_T^{\phi}(T) = \sum_{i \in T} \phi_i(N, v) = v(N) - \sum_{j \in T^c} \phi_j(N, v).$$

This exactly coincides again with (4.14) for $S = T$.

It is clear that the reductionist approach incorporates the fundamental bargaining procedure that also underlies the Core of a cooperative game. Indeed, the HM-reduced game denotes exactly the value that coalition S expects to generates if it secedes from the prevailing coalition T. Here, we incorporate that coalition S joins with the players outside T and pays all these players according to the value function ϕ.[2]

A value function is now consistent if it exactly assigns to every player the value that she would obtain in any reduced game. Hence, a consistent value exhibits no incentives to secede from the game if the value function is used as a norm in the allocation of payoffs to all players. In principle, this is an alternative formulation to describe the negotiation power of players in the determination of a binding agreement. Thus, values are assessed on Core-like principles.

Hart and Mas-Colell (1989) argued now that the HM-consistency property is essentially equivalent to the existence of a potential function. This implies that HM-consistency almost completely characterizes the Shapley value. The only missing element is a description of the value for the smallest building blocks of the reduction process, which are the two-player games. This is resolved by the following:

Definition 4.12 A value function ϕ on $\mathcal{G}$ is *standard for two-player games* if for all $(N, v) \in \mathcal{G}$ it holds that

$$\phi_i(ij, v) = v(i) + \frac{1}{2} \left[v(ij) - v(i) - v(j) \right] = v(i) + \frac{1}{2} \Delta_v(ij) \qquad (4.16)$$

for all $i, j \in N$ with $i \neq j$.

Note that the requirement that a value function is standard for two-player games is equivalent to the property stated in Proposition 4.6 for the difference system d considered by Hart and Mas-Colell (1989). This also was the conclusion of the previous section of this chapter.

The next theorem states the main insight from the HM-reductionist approach. For a proof of the assertion I again refer to the appendix of this chapter.

Theorem 4.13 *A value function ϕ on $\mathcal{G}$ that is standard for two-player games, satisfies HM-consistency if and only if ϕ is equal to the Shapley value φ.*

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[2] Other definitions of a reduced game are possible as well.

This axiomatization of the Shapley value is completely based on the theory of the cooperative potential developed in the previous sections of this chapter. In fact, this reductionist approach cannot be considered outside the setting of cooperative potential theory. It shows the very powerful instruments available through the cooperative potential to arrive at such a forceful axiomatization of the Shapley value.

As mentioned above, the reductionist approach to value theory can be extended to values other than the Shapley value. In these cases the fundamental consistency property remains unchanged, but the definition of a "reduced game" is adapted.

Sobolev (1975) used the reduced game definition introduced by Davis and Maschler (1965) to provide a reductionist approach to the *pre-nucleolus*. The so-called *DM-reduced game* in question is for all $S \subset T$ defined by

$$v_T^{\phi}(S) = \begin{cases} 0 & \text{for } S = \varnothing \\ \sum_{j \in T} \phi_j(N,v) & \text{for } S = T \\ \max_{R \subset T^c} \left(v(S \cup R) - \sum_{i \in R} \phi_i(N,v) \right) & \text{for } S \neq T, \varnothing \end{cases} \tag{4.17}$$

As is clear from this definition of the DM-reduced game, the main differences with the HM-reduced game are the application of the maximization over the various sub-coalitions and the use of the value assigned in the game (N,v) rather than $(S \cup T^c, v)$. I discuss these two differences is some detail.

First, in the DM-reduced game the coalition S is assumed to have more strategic freedom to associate with any sub-coalition in T^c. In the HM-reduced game it is assumed that all members of T^c have to be included in the deviation and paid off according to ϕ.

The second difference is how these payoffs are generated. In the DM-reduced game the members of T^c are paid according to the allocation for the whole game (N,v). Instead in the HM-reduced game these players are paid according to the value function applied to the appropriate situation.

It is hard to say which definition is more appropriate. This depends on the context in which the games are considered and the payoffs allocated. Hart and Mas-Colell (1989) argue that their HM-reduced game is appropriate for the application of value theory to allocate joint costs among several projects, departments or tasks. Each project is here to be represented by a player in the cooperative game.

As suggested in the discussion at the beginning of this section, there is yet another fruitful way to define a reduced game. Moulin (1985) introduced the *M-reduced game* by

$$v_T^{\phi}(S) = v(S \cup T^c) - \sum_{i \in T^c} \phi_i(N,v) \tag{4.18}$$

for all $S \subset T$. This definition incorporates both the clean functional form of the HM-reduced game as well as the payoff logic underlying the DM-reduced game. Indeed, coalition S is considered to seek the cooperation of the whole complement T^c, while paying them off according to the allocated values for the original game (N,v).

Moulin (1985) fully developed the reductionist approach for the M-reduced game form and arrived at a value he indicated as the *equal allocation of non-separable costs*. I will not develop this value function any further here. Instead I refer to the appropriate literature on this value.

4.4 Beyond the Cooperative Potential

In this chapter I have addressed a singular approach to the cooperative potential and the relationship of that concept with the Shapley value, fully based on Hart and Mas-Colell (1989). In this section I discuss two extensions of this approach. This first extension was developed in Calvo and Santos (1997) and applies the HM framework to other values than the Shapley value. So, a potential is related directly to a certain allocation rule or value in the sense of Corollary 4.3 rather than through the relationship of the potential function with the class of cooperative games stated in the definition of the cooperative potential.

The second approach to extending the HM framework is the relationship of potential functions and share functions. Here, a *share function* (van den Brink and van der Laan, 1998a) is a normalized value on the class of all cooperative games, assigning to every player a power index between 0 and 1. An important class of axiomatic share functions—the so-called BL-share functions—has been identified in van den Brink and van der Laan (1998a). The BL-share functions are exactly those that can be characterized using linear weight systems. For the BL-share functions one can identify certain generalized cooperative potentials, based on the linear weight systems for which these BL-share functions are characterized. This theory has been developed in van den Brink and van der Laan (2007).

4.4.1 Value-Based Potentials

Calvo and Santos (1997) identify the efficiency property imposed by Hart and Mas-Colell (1989) on their cooperative potential function Ψ as the source for its uniqueness and its relation with the Shapley value in the sense that $D\Psi = \varphi$. By abandoning the efficiency property one can extend the theory to a broad class of Shapley-like values.

We follow the definition of Ortmann (1998) of a value-based potential:

Definition 4.14 Let ϕ be some allocation rule on $\mathcal{G}$. The allocation rule ϕ *admits a potential* if there exists a function $P_\phi \colon \mathcal{G} \to \mathbb{R}$ such that for every $(N, v) \in \mathcal{G}$ and $i \in N$:

$$\phi_i(N, v) = P_\phi(N, v) - P_\phi(N - i, v). \tag{4.19}$$

This definition reverses the original definition of a potential. Instead of the summarizing of a cooperative game's "worth", here a potential is related to an allocation rule. It is clear that the Shapley value φ on $\mathcal{G}$ exactly admits the cooperative potential Ψ. Hence, the cooperative potential can also be interpreted as the "Shapley potential", based on the technical equivalence $\Psi \equiv P_\varphi$.

Our main characterization of a value-based potential is that its admittance is equivalent to the allocation rule satisfying Myerson's balanced payoff property. This result combines the main insights of Sánchez (1997), Calvo and Santos (1997) and Ortmann (1998). For a proof of this result we again refer to the appendix of this chapter.

Theorem 4.15 *Let ϕ be an allocation rule on the class of all cooperative games $\mathcal{G}$. Then the following statements are equivalent:*

(i) *ϕ admits a potential.*
(ii) *ϕ satisfies the balanced payoff property.*
(iii) *For every $(N, v) \in \mathcal{G}$ it holds that*

$$\phi(N, v) = \varphi(N, v_\phi) \tag{4.20}$$

where $v_\phi \in G^N$ is the cooperative game defined by

$$v_\phi(S) = \sum_{i \in S} \phi_i(S, v). \tag{4.21}$$

(iv) *ϕ is path-independent in the sense that for every $(N, v) \in \mathcal{G}$ and for all two permutations $\rho, \rho' : N \rightleftarrows N$ it holds that*

$$\sum_{i \in N} \phi_i\left(\mathfrak{P}(\rho, i), v\right) = \sum_{i \in N} \phi_i\left(\mathfrak{P}(\rho', i), v\right) \tag{4.22}$$

where $\mathfrak{P}(\rho, i) = \{j \in N \mid \rho(j) \leqq \rho(i)\}$ is the coalition of player $i \in N$ and all predecessors of i in ρ.

As stated, Theorem 4.15 collects the main propositions shown in the literature on cooperative potentials. In particular of interest is that an allocation rule satisfies the balanced payoff property if it admits a potential and it is based on the Shapley value φ. This again emphasizes the central role that the Shapley value has in this theory.

From Theorem 4.15 we can immediately derive that the potential functions that can be generated by these values are closely related to the cooperative potential Ψ. This is stated in the next corollary of which a proof is left to the interested reader.

Corollary 4.16 *Let ϕ be some allocation rule or value that admits a potential P_ϕ on $\mathcal{G}$. Then for every game $(N, v) \in \mathcal{G}$: $P_\phi(N, v) = \Psi(N, v_\phi)$, where v_ϕ is the ϕ-derived game as formulated in Theorem 4.15 (iii).*

Next I consider an interesting example of a value that admits a potential. This value is based on Deegan and Packel (1979), who introduce a measure to weigh the importance of certain coalitions in voting games.

Example 4.17 (The Deegan–Packel value) Following the definition of Deegan and Packel (1979) for simple games, we might generalize their formulation to the class of all cooperative games $\mathcal{G}$. Thus, the *DP-value* δ on $\mathcal{G}$ is introduced for every game (N, v) and player $i \in N$ by

$$\delta_i(N, v) = \sum_{S \subset N\colon i \in S} \frac{v(S)}{|S|} \tag{4.23}$$

Observe that $\sum_{i \in N} \delta_i(N, v) = \sum_{S \subset N} v(S)$. Hence, this DP-value is not efficient. Following the analysis above, the related cooperative game v_δ is now given by

$$v_\delta(S) = \sum_{i \in S} \delta_i(S, v) = \sum_{T \subset S} v(T).$$

It should be clear from the definition that the Harsanyi dividends of this game are given by $\Delta_{v_\delta}(S) = v(S)$. From the corollary we can now compute the DP-potential corresponding to the DP-value as

$$P_\delta(N, v) = \Psi(N, v_\delta) = \sum_{S \subset N} \frac{\Delta_{v_\delta}(S)}{|S|} = \sum_{S \subset N} \frac{v(S)}{|S|}. \tag{4.24}$$

We conclude that the DP-value is the unique allocation rule ϕ on $\mathcal{G}$ that admits a potential and satisfies the property that $\sum_{i \in N} \phi_i(N, v) = \sum_{S \subset N} v(S)$. ■

The analysis presented in this section raises the question of proper examples of allocation rules other than the Shapley value that admit such potentials. Calvo and Santos (1997) argue that the standard Myerson values based on coalitional structures belong to this class of allocation rules. Unfortunately, this is not that straightforward. Indeed, these Myerson values are based on certain coalitional structures $\Omega \subset 2^N$ for a *given* player set N, i.e., these allocation rules have domain $\mathcal{G}^N$ for a fixed N. The allocation rules in this chapter are defined on the class of all possible cooperative games $\mathcal{G}$, in which the player set is not pre-determined.

Hence, in order to establish that some Myerson value ϕ^N on $\mathcal{G}^N \times \mathfrak{W}^N$ admits a potential, one has to determine an allocation rule ϕ on the comprehensive space of *all* games with coalitional structures $\mathcal{G} \times \mathfrak{W}$ such that this rule is an amalgamate of ϕ^N over all possible player sets N. This is a non-trivial problem.

On the other hand an immediate corollary of Theorem 4.15 is an alternative characterization of the Shapley value through the notion of a potential:

Corollary 4.18 *The Shapley value φ is the unique allocation rule on $\mathcal{G}$ that is efficient and admits a potential.*

4.4.2 Share Functions and Share Potentials

The basic idea of the notion of an "allocation rule" is simply that of the division of the generated wealth $v(N)$ of a cooperative game (N, v) over the individual players in N. A natural way to allocate such wealth is to assign *shares* rather than quantities. This requires the introduction of a concept of a share function. Formally, let λ be a *share function* on $\mathcal{G}$ in the sense that for every game $(N, v) \in \mathcal{G}$ and player $i \in N$ the number $\lambda_i(N, v) \in \mathbb{R}$ is the share of that player of the total wealth generated in the game (N, v).[3] The corresponding allocation rule is now denoted by ψ^λ given by

$$\psi_i^\lambda(N, v) = \lambda_i(N, v) \cdot v(N) \tag{4.25}$$

This notion of a share function has been introduced originally in van den Brink and van der Laan (1998a). The most natural example of a share function is that of the *Shapley share function* λ^s, which is for every player $i \in N$ given by

$$\lambda_i^s(N, v) = \frac{\varphi_i(N, v)}{v(N)} \qquad \text{if } v(N) > 0. \tag{4.26}$$

Besides the Shapley share function, any properly defined allocation rule ϕ on $\mathcal{G}$ introduces a share function.

In their paper, van den Brink and van der Laan (1998a) subsequently consider a special class of share functions with certain plausible properties. In this section I will first explore this particular class of share functions and subsequently discuss the relationship of these particular share functions with the notion of the cooperative potential and its refinements.

The notion of a BL-share function puts some properties on the share function. Most of these properties are rather plausible. For this we recall that in a game (N, v) a player $i \in N$ is a *null player* if $v(S) = v(S - i)$ for all $S \subset N$ and two players $i, j \in N$ are *equipoised* if $v(S - i) = v(S - j)$ for all $S \subset N$ with $i, j \in S$.

Definition 4.19 Let $\mu \colon \mathcal{G} \to \mathbb{R}_{++}$ be some positive real-valued function. A share function λ on $\mathcal{G}$ is a *BL-share function for* μ if λ satisfies the following four properties:

(i) (*Efficient sharing*) For every game $(N, v) \in \mathcal{G}$: $\sum_{i \in N} \lambda_i(N, v) = 1$.
(ii) (μ*-additivity*) It holds that

$$\mu(N, v + w) \cdot \lambda(N, v + w) = \mu(N, v) \cdot \lambda(N, v) + \mu(N, w) \cdot \lambda(N, w) \tag{4.27}$$

for all games $(N, v), (N, w) \in \mathcal{G}$.

[3] I emphasize here that it is assumed in this definition that shares are not necessarily strictly positive. Share functions can also be defined with less generality, allowing only strictly positive shares. This seems a natural restriction for monotone games in particular.

(iii) (*Symmetry*) λ is symmetric in the sense that $\lambda_i(N, v) = \lambda_j(N, v)$ for every pair of equipoised players $i, j \in N$ in (N, v).
(iv) (*Null player property*) λ satisfies the null player property in the sense that for every null player i in (N, v) it holds that $\lambda_i(N, v) = 0$.

The definition of a BL-share function combines the standard axioms seminally considered by Shapley (1953)—namely efficiency, symmetry and the null player property—with a modified, weighted additivity concept. This μ-additivity property is a plausible extension of the regular additivity property for allocation rules.

The main question to be considered is the existence and uniqueness of such BL-share functions. For that purpose I first introduce some preliminary and auxiliary mathematical concepts. A function $f : \mathcal{G} \to \mathbb{R}$ is *additive* if for all games $(N, v), (N, w) \in \mathcal{G}$ it holds that $f(N, v + w) = f(N, v) + f(N, w)$. Hence, if λ is the BL-share function for an additive μ, μ-additivity now implies that

$$\lambda(N, v + w) = \frac{\mu(N,v)}{\mu(N,v)+\mu(N,w)}\lambda(N, v) + \frac{\mu(N,w)}{\mu(N,v)+\mu(N,w)}\lambda(N, w),$$

which represents a weighted average of the shares of the original two games. For a proof of the next characterization of BL-share functions I refer to the appendix of this chapter.

Theorem 4.20 (van den Brink and van der Laan, 1998a, Theorems 3.5 and 3.6) *Let $\mu : \mathcal{G} \to \mathbb{R}_{++}$ be some positive function. The BL-share function for μ is uniquely defined if and only if μ is additive.*

In that case the BL-share function λ for a linear function μ has the property that

$$\lambda(N, v) = \sum_{S \subset N} \frac{\Delta_v(S)\, \mu(N,u_S)}{\mu(N,v)} \lambda(N, u_S) \tag{4.28}$$

Moreover, $\lambda(N, \alpha v) = \lambda(N, v)$ for any $\alpha > 0$.

This characterization theorem clearly delineates the standard cases. It should be clear that the Shapley share function is a special case of a BL-share function.

Corollary 4.21 *Let $\mu^s : \mathcal{G} \to \mathbb{R}_{++}$ be given by $\mu^s(N, v) = v(N)$. Then the BL-share function for μ^s is exactly the Shapley share function λ^s.*

Proof We remark that for $S \subset N$: $\lambda^s(N, u_S) = \varphi(N, u_S)$ since $u_S(N) = 1$. From Theorem 4.20 we get that the BL-share function for μ^s is given by

$$\lambda(N, v) = \sum_{S \subset N} \frac{\Delta_v(S)\, \mu^s(N,u_S)}{\mu^s(N,v)} \lambda^s(N, u_S) = \sum_{S \subset N} \frac{\Delta_v(S)}{v(N)} \varphi(N, u_S) =$$
$$= \frac{\varphi\left(N, \sum_{S \subset N} \Delta_v(S)\, u_S\right)}{v(N)} = \frac{\varphi(N,v)}{v(N)} = \lambda^s(N, v)$$

which shows the assertion. ■

Another example of a share function can be developed from the allocation rule introduced by Deegan and Packel (1979).

Example 4.22 (The Deegan-Packel share function) In Example 4.17 I introduced an inefficient value δ on $\mathcal{G}$, which for every game (N, v) and player $i \in N$ is given by $\delta_i(N, v) = \sum_{S \subset N:\, i \in S} \frac{v(S)}{|S|}$. The allocation rule δ is based on the work of Deegan and Packel and has been denoted as the DP-value.

Given δ we can now construct the so-called *DP share function*, which for every game (N, v) and player $i \in N$ is formulated as

$$\lambda_i^d(N, v) = \frac{\delta_i(N, v)}{\sum_{S \subset N} v(S)} \tag{4.29}$$

This DP share function λ^d is *not* a BL-share function since it does not satisfy the null player property. Indeed, $\lambda^d(N, v) \gg 0$ for every strictly positive game $v \gg 0$. Note on the other hand that λ^d satisfies μ^d-additivity for $\mu^d(N, v) = \sum_{S \subset N} v(S)$. In fact, it can be shown that the DP share function is the unique share function that satisfies efficient sharing, symmetry, μ^d-linearity[4] and the nullifying player property.

On the other hand, according to Theorem 4.20 there has to exist a unique BL-share function λ^{μ^d} for μ^d that satisfies the null player property. I denote the share function λ^{μ^d} as the "corrected" DP-share function. For its construction, we note that the DP-share function is also μ^D-additive, where

$$\mu^D(N, v) = \frac{1}{2^{|N|-1}} \sum_{S \subset N} v(S). \tag{4.30}$$

Remark that μ^D is a linear function. Now $\lambda^{\mu^d} = \lambda^{\mu^D}$. I will use this property later to actually compute the corrected DP-share function. ■

The class of BL-share functions has particular properties. Next I explore the link of these BL-share functions to the cooperative potential. This theory is developed in van den Brink and van der Laan (2007).

We recall that for the cooperative potential Ψ it holds that

$$\sum_{i \in N} [\, \Psi(N, v) - \Psi(N - i, v)\,] = v(N).$$

This characterization of the cooperative potential can be used to formulate a potential function for the μ function on $\mathcal{G}$:

Definition 4.23 Let $\mu\colon \mathcal{G} \to \mathbb{R}$ be some real-valued function. A function $Q\colon \mathcal{G} \to \mathbb{R}$ is a *μ-share potential* if $Q(\varnothing, v) = 0$ and for all $(N, v) \in \mathcal{G}$ with $N \neq \varnothing$:

[4] The notion of μ-linearity is strengthening the μ-additivity requirement. A share function λ is *μ-linear* if for all games (N, v) and (N, w) and all real numbers a and b it holds that $\mu(N, av + bw) \cdot \lambda(N, av + bw) = a\mu(N, v)\, \lambda(N, v) + b\mu(N, w)\, \lambda(N, w)$.

$$\sum_{i \in N} [\, Q(N, v) - Q(N - i, v) \,] = \mu(N, v). \tag{4.31}$$

To formulate the main existence theorem for these share potentials some additional properties are required. A function $\mu \colon \mathcal{G} \to \mathbb{R}$ is *null player independent* if for every game (N, v) and every null player $i \in N$ it holds that $\mu(N, v) = \mu(N - i, v)$.

For a proof of Theorem 4.24 stated below I again refer to the appendix of this chapter.

Theorem 4.24 *Let $\mu \colon \mathcal{G} \to \mathbb{R}_{++}$ be additive, symmetric and null player independent.*

(a) *For the function μ there exists a unique μ-share potential Q_μ on $\mathcal{G}$.*
(b) *For the share potential Q_μ it holds that*

$$\lambda_i^\mu(N, v) = \frac{Q_\mu(N, v) - Q_\mu(N - i, v)}{\mu(N, v)}, \tag{4.32}$$

where λ^μ is the unique BL-share function for μ.
(c) *The BL-share function λ^μ for the given function μ is now characterized by*

$$\lambda^\mu(N, v) = \lambda^s(N, v_\mu),$$

where $v_\mu(S) = \mu(S, v)$ and λ^s is the Shapley share function.

From Theorem 4.24 there immediately follows a characterization of the share potential function introduced. The share potentials that are introduced here are closely related to the cooperative potential Ψ.

Corollary 4.25 *Let $\mu \colon \mathcal{G} \to \mathbb{R}_{++}$ be additive, symmetric and null player independent. The unique μ-share potential Q_μ on $\mathcal{G}$ is given by*

$$Q_\mu(N, v) = \Psi(N, v_\mu), \tag{4.33}$$

where $v_\mu(S) = \mu(S, v)$ for all coalitions $S \subset N$.

The assertion of Theorem 4.24 clearly holds for the Shapley share function λ^s. On the other, some other obvious examples of allocation rules do not satisfy the assertions stated. The next example illustrates the issues related to the previous analysis of BL-share functions. In this example I in particular illustrate this for the corrected DP-share function λ^{μ^d} with μ^d.

Example 4.26 Consider the discussion of the DP-value δ, the DP-share function λ^d, and the corrected DP-share function λ^{μ^d} conducted in Examples 4.17 and 4.22.

Let $N = \{1, 2, 3\}$ and v be given by $v(12) = 1$, $v(N) = 2$ and $v(S) = 0$ for any coalition $S \neq 12, N$. Then the Shapley value of this game is given by

$\varphi(N,v) = \frac{1}{6}(5,5,2)$ and the Shapley share vector for this game is thus determined to be $\lambda^s(N,v) = \frac{1}{2}\varphi(N,v) = \frac{1}{12}(5,5,2)$. Also the DP-value is given by $\delta(N,v) = \frac{1}{6}(7,7,4)$. It is easy to compute that the DP-share function is now given by $\lambda^d(N,v) = \frac{1}{18}(7,7,4)$. Furthermore, $\mu^d(12,v) = 1$, $\mu^d(N,v) = 3$ and $\mu^d(S,v) = 0$ otherwise. This also defines the derived game $v^d = v_{\mu^d}$ on N.

Remark now that $\lambda^s(N,v^d) = \frac{1}{3}\varphi(N,v^d) = \frac{1}{3}\cdot\frac{1}{6}(7,7,4) = \frac{1}{18}(7,7,4) = \lambda^d(N,v)$. The main question addressed in this example is whether the Deegan-Packel BL-share function λ^{μ^d} leads to a very different result.

This is indeed the case. Namely, from the given data, one computes that $\mu^D(12,v) = \frac{1}{2}$, $\mu^D(N,v) = \frac{3}{4}$ and $\mu^D(S,v) = 0$ for all $S \neq 12, N$. From this it follows that the corrected DP-share function is given by $\lambda^{\mu^d}(N,v) = \lambda^{\mu^D}(N,v) = \lambda^s(N,v_{\mu^D}) = \frac{1}{9}(4,4,1)$.

The difference between the DP-share function $\lambda^d(N,v) = \lambda^s(N,v^d)$ and the corrected DP-share function $\lambda^{\mu^d}(N,v)$ can be explained by pointing out that the function μ^d does not satisfy the null player independence property that is required in Theorem 4.24. Indeed, removing a null player would remove the double counting of certain coalitional outputs in μ^d. On the other hand, the function μ^D satisfies the null player independence property due to its normalization. ■

4.5 Appendix: Proofs of the Main Theorems

Proof of Theorem 4.2

We can rewrite (4.3) as

$$\Psi(N,v) = \frac{1}{|N|}\left[v(N) + \sum_{i\in N}\Psi(N-i,v)\right] \tag{4.34}$$

Starting with $\Psi(\varnothing,v) = 0$, it determines $\Psi(N,v)$ recursively. This proves the existence of a potential function Ψ. Furthermore, it is easy to see that (4.3) determines the function Ψ uniquely. This shows assertion (a).

We first prove the second formulation in (b). Consider $\Psi(N,v) = \sum_{S\subset N}\frac{\Delta_v(S)}{|S|}$. Then it can be deduced that

$$D_i\Psi(N,v) = \sum_{S\subset N}\frac{\Delta_v(S)}{|S|} - \sum_{T\subset N-i}\frac{\Delta_v(T)}{|T|} = \sum_{S:\, i\in S}\frac{\Delta_v(S)}{|S|},$$

and consequently

$$\sum_{i\in N} D_i\Psi(N,v) = \sum_{i\in N}\sum_{S:\, i\in S}\frac{\Delta_v(S)}{|S|} = \sum_{S\subset N}\Delta_v(S) = v(N).$$

This shows indeed that the provided formulation satisfies the definition of the potential function.

The third formulation is a direct consequence of the second formulation and the formulation of the MLE of a game provided in (1.25). Indeed,

$$E_v(t,\dots,t) = \sum_{S\subset N} \Delta_v(S) t^{|S|} \tag{4.35}$$

and, so,

$$\int_0^1 \frac{E_v(t,\dots,t)}{t}\,dt = \int_0^1 \sum_{S\subset N} \Delta_v(S) t^{|S|-1} dt = \sum_{S\subset N} \frac{\Delta_v(S)}{|S|} t^{|S|}\Bigg|_0^1 =$$
$$= \sum_{S\subset N} \frac{\Delta_v(S)}{|S|} = \Psi(N,v)$$

For the final formulation we remark that a coalition $S \subset N$ of size $s = |S|$ is chosen with probability

$$p_s^n = \frac{(s-1)!\,(n-s)!}{n!} = \frac{1}{\binom{n}{s}}.$$

This corresponds exactly to putting players in random order and form coalitions according to this order.[5] Now we can write

$$\sum_{S\subset N} \frac{(|S|-1)!\,(n-|S|)!}{n!} v(S) = \sum_{S\subset N} p_s^n\, v(S).$$

It is easy to check that this is indeed the unique potential $\Psi(N,v)$. This shows the first formulation listed in Theorem 4.2(b).

The fourth and final formulation listed in Theorem 4.2(b) now follows immediately from the above as well.

Proof of Theorem 4.7

Let (N,v) be given. We introduce some short hand notation. For every coalition $S \subset N$ by $\Psi(S)$ we now refer to the potential $\Psi(S,v)$.

Assume that the allocation rule ϕ already has been determined for all proper subgames (S,v) of (N,v) with $S \subsetneq N$ and that in fact it is given by

---

[5] This is the model of coalition formation based on the notion of a Weber string as pursued in Chapter 2 of these lecture notes.

$$\phi_i(S) = \Psi(S) - \Psi(S - i).$$

Note that $\Psi(i) = v(i)$ and therefore $\phi_i(i) = v(i)$, which corresponds to the initial condition (4.12) for the recursive definition of ϕ.

Applying the definition of ϕ to all coalitions of size $n-1$ we get that

$$\begin{aligned} d^{\phi}_{ij} &= \phi_i(N-j) - \phi_j(N-i) = \\ &= \left[\,\Psi(N-j) - \Psi(N-ij)\right] - \left[\,\Psi(N-i) - \Psi(N-ij)\right] = \\ &= \Psi(N-j) - \Psi(N-i). \end{aligned}$$

Hence, the difference system d^{ϕ} is indeed compatible.

This in turn implies through Proposition 4.6 that ϕ is well-defined as well and given by

$$\phi_i(N) = \frac{1}{n}\left[\, v(N) + \sum_{j \neq i} d^{\phi}_{ij} \,\right] = \frac{1}{n}\left[\, v(N) + \sum_{j \neq i} \left(\Psi(N-j) - \Psi(N-i)\right) \,\right].$$

Hence, we derive that

$$\phi_i(N) + \Psi(N-i) = \frac{1}{n}\left[\, v(N) + \sum_{j \in N} \Psi(N-j) \,\right] = \Psi(N)$$

This leads to the desired conclusion that $\phi = \varphi$.

Proof of Theorem 4.13

We follow the proof developed in Hart and Mas-Colell (1989, pp. 597–599).

This proof is based on the simplification of the HM-consistency requirement for a value function. The next intermediary result states this insight.

Lemma 4.27 *A value function ϕ on $\mathcal{G}$ is HM-consistent if and only if (4.15) is satisfied for all games (N,v) and all coalitions $T \subset N$ with $|T^c| = 1$.*

Proof We only have to show that if (4.15) is satisfied for all games (N,v) and all coalitions $T \subset N$ with $|T^c| = 1$, then (4.15) is satisfies for all coalitions T.

Now let $T \subset N$ with $|T^c| = k$. Assume the asserted hypothesis that for all games (N,v)

$$\phi_i(N,v) = \phi_i\left(N-j, v^{\phi}_{N-j}\right)$$

is satisfied for all $i,j \in N$ with $i \neq j$.

Returning to the definition of the HM-reduced game, we note that for (N, v) and $i, j \in N$ with $i \neq j$ it holds that for every $S \subset N - ij = N \setminus \{i, j\}$:

$$\begin{aligned}\left(v^{\phi}_{N-i}\right)^{\phi}_{N-ij}(S) &= v^{\phi}_{N-i}(S+j) - \phi_j\left(S+j, v^{\phi}_{N-i}\right) \\ &= v(S+ij) - \phi_i(S+ij, v) - \phi_j(S+ij, v) \\ &= v^{\phi}_{N-ij}(S),\end{aligned}$$

where in the second equality we use the hypothesis that

$$\phi_j(S+ij, v) = \phi_j\left(S+j, v^{\phi}_{S-i}\right) = \phi_j\left(S+j, v^{\phi}_{N-i}\right).$$

Hence, from this construction we deduce from the hypothesis that for all games (N, v) and all distinct players $i, j, h \in N$

$$\begin{aligned}\phi_i(N, v) &= \phi_i\left(N-j, v^{\phi}_{N-j}\right) \\ &= \phi_i\left(N-jh, \left(v^{\phi}_{N-j}\right)^{\phi}_{N-jh}\right) = \phi_i\left(N-jh, v^{\phi}_{N-jh}\right)\end{aligned}$$

Hence, by repeated application of this deduction, following the enumeration of the k players in the coalition T^c, we derive immediately the assertion. This completes the proof of the lemma. ■

Next we turn to the actual proof of the assertion stated in Theorem 4.13. Obviously the Shapley value is standard for two-player games. We now show that the Shapley value is indeed HM-consistent.

We use the potential to show this property. Let (N, v) be given and $i \in N$. Now write $v_{-i} := v^{\varphi}_{N-i}$ as the HM-reduced for $N - i = N \setminus \{i\}$ and the Shapley value φ. Since φ is efficient we can write for $S \subset N - i$:

$$\begin{aligned}v_{-i}(S) = v(S+i) - \varphi_i(S+i, v) &= \sum_{j \in S} \varphi_j(S+i, v) = \\ &= \sum_{j \in S} \left[\, \Psi(S+i, v) - \Psi(S+i-j, v) \,\right] = \sum_{j \in S} D_j\, \Psi(S+i, v).\end{aligned}$$

By Theorem 4.2 applied to $(N - i, v_{-i})$ and all of its subgames, we have a *unique* determination of its potential. We can in fact deduce that

$$\Psi(S, v_{-i}) = \Psi(S+i, v) - \Psi(i, v) = \Psi(S+i, v) - v(i). \tag{4.36}$$

Indeed, with the above if follows that

$$\Psi(j, v_{-i}) = v_{-i}(j) = D_j\, \Psi(ij, v) = \Psi(ij, v) - \Psi(i, v).$$

Now assume that the formula holds for coalition S. Then for $S+j$ it holds that

$$\begin{aligned}
v_{-i}(S+j) &= D_j\,\Psi(S+j,v_{-i}) + \sum_{h\in S} D_h\,\Psi(S+j,v_{-i})\\
&= (|S|+1)\,\Psi(S+j,v_{-i}) - \Psi(S,v_{-i}) - \sum_{h\in S}\Psi(S+j-h,v_{-i})\\
&= (|S|+1)\,\Psi(S+j,v_{-i}) - \Psi(S+i,v)\\
&\quad - \sum_{h\in S}\Psi(S+ij-h,v) + (|S|+1)\,v(i)
\end{aligned}$$

On the other hand, from the above, it also follows that

$$v_{-i}(S+j) = \Psi(S+ij,v) - \Psi(S+i,v) + \sum_{h\in S}\big[\,\Psi(S+ij,v) - \Psi(S+ij-h,v)\big]$$

Equating the two previous expressions we arrive at

$$(|S|+1)\,\Psi(S+j,v_{-i}) + (|S|+1)\,v(i) = (|S|+1)\,\Psi(S+ij,v)$$

and, therefore, $\Psi(S+j,v_{-i}) = \Psi(S+ij,v) - v(i)$. This proves the claim.

Now we can compute that

$$\begin{aligned}
\varphi_j(N-i,v_{-i}) &= \Psi(N-i,v_{-i}) - \Psi(N-ij,v_{-i}) =\\
&= \Psi(N,v) - \Psi(N-j,v) = \varphi_j(N,v)
\end{aligned}$$

Together with Lemma 4.27 this completes the proof that the Shapley value is indeed HM-consistent.

It remains to show that if a value function ϕ is HM-consistent as well as standard for two-player games, then it necessarily is the Shapley value φ.

We first show that ϕ is efficient if it satisfies HM-consistency and is standard for two-player games. We apply induction on n to show this. It trivially holds by the standard-for-two-player-games property for $n=2$. Next assume by induction that ϕ is efficient for all games (N,v) with $2 \leqq |N| \leqq n-1$. Let (N,v) be such that $|N| = n > 2$ and let $i \in N$. Then by HM-consistency and Lemma 4.27

$$\sum_{j\in N}\phi_j(N,v) = \sum_{j\in N-i}\phi_j(N-i,v_{-i}) + \phi_i(N,v)$$

where $v_{-i} = v^{\phi}_{N-i}$ as before. By the induction hypothesis it now follows that

$$\sum_{j\in N}\phi_j(N,v) = v_{-i}(N-i) + \phi_i(N,v) = v(N)$$

by definition of v_{-i}. This shows that ϕ is indeed efficient for $n \geqq 2$.

Finally for $n = 1$ we have to show that $\phi_i(i, v) = v(i)$. Let $v(i) = c$ and $j \neq i$. Now consider the game $(ij, \bar{v})$ given by $\bar{v}(i) = \bar{v}(ij) = c$ and $\bar{v}(j) = 0$. Then by ϕ being the standard for two-player games it follows that $\phi_i(ij, \bar{v}) = c$ and $\phi_j(ij, \bar{v}) = 0$. Hence, $\bar{v}_{-j}(i) = c - 0 = c = v(i)$ and $c = \phi_i(ij, \bar{v}) = \phi_i(i, \bar{v}_{-j}) = \phi_i(i, v)$ by HM-consistency. This shows that ϕ is efficient for $n = 1$ as well.

Next we show that ϕ defines in fact a potential. For that purpose introduce a function Q on the class of all games with at most two players given by

$$
\begin{aligned}
Q(\varnothing, v) &= 0 \\
Q(i, v) &= v(i) \\
Q(ij, v) &= \frac{1}{2}\left[\, v(i) + v(j) + v(ij) \,\right]
\end{aligned}
$$

It is straightforward to check that by ϕ being standard for two-player games for all games (N, v) with $|N| = n = 1, 2$ we have that

$$\phi_i(N, v) = Q(N, v) - Q(N - i, v), \quad i \in N. \tag{4.37}$$

We now show that Q can be extended to all games $(N, v) \in \mathcal{G}$ such that (4.37) holds. Together with the efficiency of ϕ—that is shown above—this implies that Q is in fact the potential Ψ and that ϕ is the Shapley value on $\mathcal{G}$.

We once more apply induction. Let (N, v) be given with $n \geqq 3$ and assume that Q has been defined and satisfies (4.37) for all games with at most $n - 1$ players.

We show that

$$\phi_i(N, v) + Q(N - i, v)$$

is the same for all $i \in N$ and chosen to be equal to $Q(N, v)$. Let $i, j \in N$ with $i \neq j$. Select $k \in N$ with $k \notin \{i, j\}$. Then by consistency and (4.37) for $n - 1$:

$$
\begin{aligned}
\phi_i(N, v) - \phi_j(N, v) &= \phi_i(N - k, v_{-k}) - \phi_j(N - k, v_{-k}) \\
&= \left[\, Q(N - k, v_{-k}) - Q(N - ik, v_{-k}) \,\right] \\
&\quad - \left[\, Q(N - k, v_{-k}) - Q(N - jk, v_{-k}) \,\right] \\
&= \left[\, Q(N - jk, v_{-k}) - Q(N - ijk, v_{-k}) \,\right] \\
&\quad - \left[\, Q(N - ik, v_{-k}) - Q(N - ijk, v_{-k}) \,\right] \\
&= \phi_i(N - jk, v_{-k}) - \phi_j(N - ik, v_{-k}) \\
&= \phi_i(N - j, v_{-k}) - \phi_j(N - i, v_{-k}) \\
&= \left[\, Q(N - j, v) - Q(N - ij, v) \,\right] \\
&\quad - \left[\, Q(N - i, v) - Q(N - ij, v) \,\right] \\
&= Q(N - j, v) - Q(N - i, v)
\end{aligned}
$$

where we used HM-consistency and (4.37) for $n-2$ as well as $n-1$ in the intermediate steps in the deduction.

This deduction completes the proof of the assertion that ϕ and Q are related in this fashion and that, therefore, $\phi = \varphi$ and $Q = \Psi$.

Proof of Theorem 4.15

(i) implies (ii)
Let P be a potential of ϕ in the sense that $\phi_i(N,v) = P(N,v) - P(N-i,v)$. Then for all $i,j \in N$ with $i \neq j$:

$$\begin{aligned}\phi_i(N,v) - \phi_i(N-j,v) &= [\,P(N,v) - P(N-i,v)\,] - \big[\,P(N-j,v) - P(N-ij,v)\,\big]\\ &= \big[\,P(N,v) - P(N-j,v)\,\big] - \big[\,P(N-i,v) - P(N-ij,v)\,\big]\\ &= \phi_j(N,v) - \phi_j(N-i,v)\end{aligned}$$

(ii) implies (iii)
Suppose that for all S and $i,j \in S$: $\phi_i(S,v) - \phi_i(S-j,v) = d_S$. To prove the assertion we apply induction on the size of the player set N.

First, assume that $|N| = 1$, i.e., $N = \{i\}$. Then by definition $\phi_i(N,v) = v_\phi(\{i\})$. Hence, by efficiency of the Shapley value we get that

$$\varphi_i(N,v_\phi) = v_\phi(\{i\}) = \phi_i(N,v).$$

Next, suppose as the induction hypothesis that $\phi(S,v) = \varphi(S,v_\phi)$ for all $S \subsetneq N$. We remark that the Shapley value also satisfies the balanced payoff property. Hence, for any $i,j \in N$:

$$\begin{aligned}\phi_i(N,v) - \varphi_i(N,v_\phi) &= d_N + \phi_i(N-j,v) - \varphi_i(N,v_\phi)\\ &= d_N + \varphi_i(N-j,v_\phi) - \varphi_i(N,v_\phi)\\ &= d_N + \varphi_j(N-i,v_\phi) - \varphi_j(N,v_\phi)\\ &= d_N + \phi_j(N-i,v) - \varphi_j(N,v_\phi)\\ &= \phi_j(N,v) - \varphi_j(N,v_\phi) = k,\end{aligned}$$

where k is some given constant for (N,v). Now,

$$|N| \cdot k = \sum_{h \in N} \big[\,\phi_h(N,v) - \varphi_h(N,v_\phi)\,\big] = v_\phi(N) - v_\phi(N) = 0$$

by efficiency of the Shapley value φ and (ii). This implies that $k = 0$, thus showing assertion (iii).

(iii) implies (i)
Suppose that $\phi(N, v) = \varphi(N, v_\phi)$ for all $(N, v) \in \mathcal{G}$. Define $P\colon \mathcal{G} \to \mathbb{R}$ by $P(N, v) = \Psi(N, v_\phi)$, where Ψ is the cooperative (Shapley) potential.

Then for every $i \in N$:

$$P(N, v) - P(N - i, v) = \Psi(N, v_\phi) - \Psi(N - i, v_\phi) = \varphi_i(N, v_\phi) = \phi_i(N, v).$$

This shows the assertion.

(i) implies (iv)
Let P be a potential for ϕ. Now for some permutation $\rho\colon N \rightleftarrows N$ we get that

$$\sum_{i \in N} \phi_i(\mathfrak{P}(\rho, i), v) = \sum_{i \in N} [\, P(\mathfrak{P}(\rho, i), v) - P(\mathfrak{P}(\rho, i) - i, v)\,] = P(N, v) - P(\varnothing, v)$$

This is indeed independent of the choice of the selected permutation ρ. Hence, the assertion holds.

(iv) implies (ii)
Let $i, j \in N$ with $i \neq j$ and let $\overline{\rho}\colon N - ij \rightleftarrows N - ij$ be some given permutation on $N - ij = N \setminus \{i, j\}$. Now define $\rho^1 = (\overline{\rho}, i, j)$ and $\rho^2 = (\overline{\rho}, j, i)$. Then by (iv):

$$\sum_{h \in N} \phi_h(\mathfrak{P}(\rho^1, h), v) = \sum_{h \neq i,j} \phi_h(\mathfrak{P}(\overline{\rho}, h), v) + \phi_i(N - j, v) + \phi_j(N, v)$$

and

$$\sum_{h \in N} \phi_h(\mathfrak{P}(\rho^2, h), v) = \sum_{h \neq i,j} \phi_h(\mathfrak{P}(\overline{\rho}, h), v) + \phi_j(N - i, v) + \phi_i(N, v).$$

Equating these two expressions implies that

$$\phi_i(N - j, v) + \phi_j(N, v) = \phi_j(N - i, v) + \phi_i(N, v)$$

implying that the payoffs for players i and j are balanced.

Proof of Theorem 4.20

I first show the first assertion:

Only if Let λ satisfy efficient sharing and μ-additivity. Now, from μ-additivity: $\mu(N, v + w) \sum_{i=1}^{n} \lambda_i(N, v + w) = \mu(N, v) \sum_{i=1}^{n} \lambda_i(N, v) + \mu(N, w) \sum_{i=1}^{n} \lambda_i(N, w)$ for v, w and i arbitrary. Efficient sharing now implies that $\mu(N, v + w) = \mu(N, v) + \mu(N, w)$, which shows the assertion that μ is additive.

If Let μ be additive. I first show that there is at most one BL-share function for μ. For that purpose let λ be a BL-share function for μ. For a unanimity game (N, u_S) with $S \subset N$, two players i and j are equipoised if they both belong to S, while all players outside S are null players. Hence, from symmetry, the null player property and efficient sharing it follows that for any $\alpha > 0$:

$$\lambda_i(N, \alpha u_S) = \begin{cases} \frac{1}{|S|} & \text{if} i \in S \\ 0 & \text{if} i \notin S \end{cases} \tag{4.38}$$

Now let $(N, v) \in \mathcal{G}$ and write

$$v = \sum_{S \in \mathcal{T}_v^+} \Delta_v(S)\, u_S - \sum_{S \in \mathcal{T}_v^-} (-\Delta_v(S))\, u_S$$

where $\mathcal{T}_v^+ = \{S \subset N \mid \Delta_v(S) > 0\}$ and $\mathcal{T}_v^- = \{S \subset N \mid \Delta_v(S) < 0\}$. Since μ is positive, it follows from repeated application of μ-additivity that

$$\begin{aligned} \mu(N, v)\, \lambda(N, v) = & \sum_{S \in \mathcal{T}_v^+} \mu(N, \Delta_v(S)\, u_S) \cdot \lambda(N, \Delta_v(S)\, u_S) - \\ & - \sum_{S \in \mathcal{T}_v^-} \mu(N, -\Delta_v(S)\, u_S) \cdot \lambda(N, -\Delta_v(S)\, u_S) \end{aligned} \tag{4.39}$$

and, with the above, it follows that $\lambda(N, v)$ is indeed uniquely defined.

It remains to be shown that the uniquely defined λ satisfies the four introduced properties. Now, from additivity of μ we conclude that

$$\mu(N, v) = \sum_{S \in \mathcal{T}_v^+} \mu(N, \Delta_v(S)\, u_S) - \sum_{S \in \mathcal{T}_v^-} \mu(N, \Delta_v(S)\, u_S).$$

With (4.38) and (4.39) it now follows that $\sum_{j=1}^{n} \lambda_j(N, v) = 1$, which shows efficient sharing. Second, observe that a null player in (N, v) is a null player in u_S for any set S with non-zero dividend $\Delta_v(S) \neq 0$. Hence, again from (4.38) and (4.39) and the positivity of μ, it follows that λ satisfies the null player property. Third, if i and j are equipoised,then $\Delta_v(i) = \Delta_v(j)$, whereas for each $S \subset N$ with $\Delta_v(S) \neq 0$, i and j are either both in S or both not in S. Hence, from the above and positivity of μ, it follows that λ is symmetric. Fourth, for any two games (N, v) and (N, w):

$$v + w = \sum_{S \subset N} (\, \Delta_v(S) + \Delta_w(S)\,)\; u_S$$

Together with (4.39) and additivity of μ it can be concluded that λ is μ-additive. This shows the first part of the assertion in Theorem 4.20.

Next we actually compute the unique BL-share function for a linear μ. From (4.38) and the linearity of μ we can simplify (4.39) to

$$\mu(N,v)\,\lambda(N,v) = \sum_{S\in\mathcal{T}_v^+} \Delta_v(S)\,\mu(N,u_S)\,\lambda(N,u_S) - \\ - \sum_{S\in\mathcal{T}_v^-} (-\Delta_v(S))\,\mu(N,u_S)\,\lambda(N,u_S) \\ = \sum_{S\subset N} \Delta_v(S)\,\mu(N,u_S)\,\lambda(N,u_S) \tag{4.40}$$

Since $\mu(N,v) > 0$, the formula stated in Theorem 4.20 now follows immediately.

Next, note that

$$\alpha\, v = \sum_{S\subset N} \Delta_{\alpha v}(S)\,u_S = \sum_{S\subset N} \alpha\,\Delta_v(S)\,u_S$$

Now from (4.40)

$$\mu(N,\alpha v)\,\lambda(N,\alpha v) = \sum_{S\subset N} \alpha\,\Delta_v(S)\,\mu(N,u_S)\lambda(N,u_S).$$

Since μ is linear, $\mu(\alpha v) = \alpha\,\mu(v)$ and so,

$$\lambda(\alpha v) = \frac{\sum_{S\subset N} \alpha\,\Delta_v(S)\,\mu(N,u_S)\lambda(N,u_S)}{\mu(\alpha v)} \\ = \frac{\sum_{S\subset N} \Delta_v(S)\,\mu(N,u_S)\lambda(N,u_S)}{\mu(v)} = \lambda(N,v),$$

which proves the last assertion in Theorem 4.20.

Proof of Theorem 4.24

To show assertion (a), note that with $Q(\varnothing,v) = 0$ a potential function Q is completely determined by rewriting the defining Equation (4.31) as

$$Q(N,v) = \tfrac{1}{n}\left(\mu(N,v) + \sum_{i\in N} Q(N-i,v)\right)$$

This equation in fact determines Q recursively and, thus, uniquely. This shows (a).

To show assertion (b), we define λ by

$$\lambda_i(N,v) = \frac{Q_\mu(N,v) - Q_\mu(N-i,v)}{\mu(N,v)}.$$

To prove that $\lambda = \lambda^{\mu}$ we only have to show that λ satisfies the conditions listed in Theorem 4.20, namely the null player property, symmetry, efficient sharing and μ-additivity. Below we use induction to prove all properties except efficient sharing.

Efficient sharing This follows immediately from the definition of λ and (4.31).

μ-additivity For λ to be μ-additive it only has to be shown that the corresponding share potential Q_μ is additive. For $|N| = 1$ it is obvious that

$$Q_\mu(N, v+w) = \mu(N, v+w) = \mu(N, v) + \mu(N, w) = Q_\mu(N, v) + Q_\mu(N, w)$$

implying the assertion.

Proceed by induction and assume that λ satisfies μ-additivity for $|N| \leqq k$. Hence, for $|N| \leqq k$ it holds that $Q_\mu(N, v+w) = Q_\mu(N, v) + Q_\mu(N, w)$. Then for N with $|N| = k+1$ we get by induction on $N - j, j \in N$, that

$$\begin{aligned} Q_\mu(N, v+w) &= \tfrac{1}{n}\left(\mu(N, v+w) + \sum_{j\in N} Q_\mu(N-j, v+w)\right) \\ &= \tfrac{1}{n}\left(\mu(N, v) + \mu(N, w)\right) + \\ &\qquad + \tfrac{1}{n}\sum_{j\in N}\left(Q_\mu(N-j, v) + Q_\mu(N-j, w)\right) \\ &= Q_\mu(N, v) + Q_\mu(N, w) \end{aligned}$$

From this μ-additivity for λ follows immediately.

Null player property Let $i \in N$ be a null player in (N, v). First, assume $N = \{i\}$. Then $v = 0$ and so $Q_\mu(N, v) = 0$ and

$$Q_\mu(N, v) - Q_\mu(N - i, v) = Q_\mu(\{i\}, v) = \mu(N, v) = 0.$$

Hence, $\lambda_i(N, v) = 0$.

Next, proceed by induction and assume that for any game (N', v) with $|N'| \leqq k$ and any null player $j \in N'$: $\lambda_i(N', v) = 0$. Let $|N| = k+1$. Using the induction hypothesis on $N - j$ for any $j \in N$ we arrive at

$$\begin{aligned} nDQ_{\mu,i}(N, v) &= DQ_{\mu,i}(N, v) + \sum_{j\neq i} DQ_{\mu,i}(N, v) \\ &= DQ_{\mu,i}(N, v) + \sum_{j\neq i}\left(DQ_{\mu,i}(N, v) - DQ_{\mu,i}(N-j, v)\right) \end{aligned}$$

$$
\begin{aligned}
&= Q_\mu(N,v) - Q_\mu(N-i,v) + \\
&\qquad + \sum_{j \neq i} \big(Q_\mu(N,v) - Q_\mu(N-j,v) \big) + \\
&\qquad - \sum_{j \neq i} \big(Q_\mu(N-i,v) - Q_\mu(N-ij,v) \big) \\
&= \mu(N,v) - \mu(N-i,v).
\end{aligned}
$$

By the fact that i is a null player in (N,v) and null player independency of μ, we conclude that

$$DQ_{\mu,i}(N,v) = \tfrac{1}{n} \left(\mu(N,v) - \mu(N-i,v) \right) = 0,$$

and therefore that $\lambda_i(N,v) = 0$. This shows the null player property for λ.

Symmetry Again we use the method of induction to show the symmetry property for λ. Let $i,j \in N$ be two equipoised players in (N,v). For $N = \{i,j\} = ij$ it follows with the symmetry of μ that $Q_\mu(N-i,v) = \mu(\{j\},v) = \mu(\{i\},v) = Q_\mu(N-j,v)$. Hence, $\lambda_i(ij,v) = \lambda_j(ij,v)$.

Proceed by induction and assume that for every game (N',v) with $|N'| \leqq k$ the share function λ is symmetric. Now assume that $|N| = k+1$. Using symmetry of μ and the induction hypothesis for $N-h$, $h \in N$, we obtain that

$$
\begin{aligned}
Q_\mu(N-i,v) &= \tfrac{1}{n-1} \Big(\mu(N-i,v) - \sum h \neq i Q_\mu(N-ih,v) \Big) \\
&= \tfrac{1}{n-1} \Big(\mu(N-j,v) - \sum h \neq j Q_\mu(N-jh,v) \Big) \\
&= Q_\mu(N-j,v).
\end{aligned}
$$

Thus, for symmetric players $i,j \in N$ we arrive at

$$DQ_{\mu,i}(N,v) = Q_\mu(N,v) - Q_\mu(N-i,v) = DQ_{\mu,j}(N,v).$$

This implies in turn that $\lambda_i(N,v) = \lambda_j(N,v)$, showing the assertion.

From the above it now follows that λ indeed is the unique BL-share function for μ. This shows therefore assertion (b).

Finally, we show assertion (c). Let $(N,v) \in \mathcal{G}$. Now since $\mu^s(N,v_\mu) = v_\mu(N) = \mu(N,v) > 0$, it follows that (N,v_μ) indeed is a regular economic game. Next define $\lambda(N,v) = \lambda^s(N,v_\mu)$. We show that λ is in fact a share function and satisfies the properties of a BL-share function. With Theorem 4.20 this shows that $\lambda = \lambda^\mu$ is in fact the unique BL-share function with μ.

That λ satisfies efficient sharing follows immediately from the fact that the Shapley share function λ^s satisfies it. Also, if $i \in N$ is a null player in (N,v), then i is also a null player in (N,v_μ); thus, $\lambda_i(N,v) = \lambda^s(N,v_\mu) = 0$. Furthermore, let $i,j \in N$

be two equipoised players in (N, v). Then symmetry of μ implies that i and j are symmetric in (N, v_μ). Thus, symmetry of λ follows immediately.

To show μ-additivity of λ, we note that additivity of μ implies that $(v+w)_\mu = v_\mu + w_\mu$. Hence, we obtain

$$\begin{aligned}\mu(N, v+w)\cdot\lambda(N, v+w) &= \mu(N, v+w)\cdot\lambda^s(N, (v+w)_\mu)\\ &= \mu(N, v+w)\cdot\lambda^s(N, v_\mu + w_\mu) =\end{aligned}$$

$$\begin{aligned}&= \mu(N, v+w)\left(\frac{\mu^s(N, v_\mu)\lambda^s(N, v_\mu) + \mu^s(N, w_\mu)\lambda^s(N, w_\mu)}{\mu^s(N, (v+w)_\mu)}\right)\\ &= \mu(N, v+w)\left(\frac{v_\mu(N)\lambda(N, v) + w_\mu(N)\lambda(N, w)}{(v+w)_\mu(N)}\right)\\ &= \mu(N, v+w)\left(\frac{\mu(N, v)\lambda(N, v) + \mu(N, w)\lambda(N, w)}{\mu(N, v+w)}\right)\\ &= \mu(N, v)\lambda(N, v) + \mu(N, w)\lambda(N, w).\end{aligned}$$

This indeed shows μ-additivity of λ, thus showing the assertion.

4.6 Problems

Problem 4.1 Consider the cooperative potential $\Psi\colon \mathcal{G} \to \mathbb{R}$.

(a) Consider the extended cooperative game represented by the characteristic function[6] $v_1\colon 2^{\mathbb{N}} \to \mathbb{R}$ with the property that $\Psi(S, v_1) = 1$ for all $S \neq \varnothing$. Construct the characteristic function v_1 on any player set N from the potential values given.

(b) Let $v \in \mathcal{G}$ be any cooperative game and let N be any finite player set. Prove that the game v on N, i.e., the characteristic function $v\colon 2^N \to \mathbb{R}$, can be recovered from the potential values $\{\Psi(S, v) \mid S \subset N,\ S \neq \varnothing\}$.

Problem 4.2 Consider any cooperative game $(N, v) \in \mathcal{G}$. Show that (N, v) can be decomposed using potential values as follows:

$$v = \sum_{S\subset N} \Psi(S, v)\, p^v_S \tag{4.41}$$

[6] Here the notion "extended" refers to the fact that this characteristic function is defined for arbitrary finite players sets.

where $p_S^v \in \mathcal{G}^N$ is the "potential decomposition" game corresponding to the coalition $S \subset N$ and the game v.

(a) Give a complete description of the game p_S^v and provide an interpretation of this decomposition game. Is there any clear relationship of these games to the unanimity basis games?
(b) Are these potential decomposition games independent of the chosen game v?
(c) Does the collection of the games $\{p_S^v \mid S \subset N\}$ form a basis of the linear space of cooperative games $\mathcal{G}^N$?

Problem 4.3 Consider a monotone function $f\colon \mathbb{N} \to \mathbb{N}$ with $f(n) \leqq f(n+1)$ for all $n \in \mathbb{N}$. Now assume that $v \in G$ is determined by $\Psi(S, v) = f(|S|)$. Give a complete expression of $v(S)$ for arbitrary coalitions S. What can be said about the Shapley value of v? Try to be complete in your analysis.

Problem 4.4 Construct a proof of Corollary 4.9.

Problem 4.5 Sprumont (1990) has introduced a recursive formulation of the Shapley value. Namely, the Shapley value φ on $\mathcal{G}$ can also be defined recursively by

$$\varphi_i(N, v) = v(\{i\}) \qquad \text{for } N = \{i\}$$

$$\varphi_i(N, v) = \frac{1}{n}\left[v(N) - v(N - i) + \sum_{j \neq i} \varphi_i(N - j, v) \right]$$

for every $(N, v) \in \mathcal{G}$ and $i \in N$.
Show that the Shapley value indeed is defined by the formulation.

Problem 4.6 An allocation rule ϕ on $\mathcal{G}$ satisfies the *Sprumont property* (Sprumont, 1990) if it holds that for every $(N, v) \in \mathcal{G}$ with $|N| \geqq 2$ and $i \in N$:

$$\phi_i(N, v) = \frac{1}{n}\left[v_\phi(N) - v_\phi(N - i) + \sum_{j \neq i} \phi_i(N - j, v) \right] \tag{4.42}$$

where $v_\phi(S) = \sum_{i \in S} \phi_i(S, v)$.

(a) Show that if the value ϕ satisfies the Sprumont property, then ϕ also satisfies the balanced payoffs property.
(b) Does the converse of the assertion stated in (a) also hold? Provide a counter example or otherwise a proof.
(c) Show that the shapley value φ is the unique efficient value that satisfies the Sprumont property.

Problem 4.7 Consider the *marginal value* on $\mathcal{G}$. This is the value m which for every $(N, v) \in \mathcal{G}$ and $i \in N$ is defined by

$$m_i(N, v) = v(N) - v(N - i).$$

Show the following properties of this marginal value.

(a) The marginal value satisfies the balanced payoff property.
(b) The potential P_0 on $\mathcal{G}$ given by $P_0(N, v) = v(N)$ is the potential corresponding to the marginal value m.
(c) Show that the derived game v_m is given by

$$v_m(S) = \sum_{i \in S} m_i(S, v) = |S| \cdot v(S) - \sum_{i \in S} v(S - i).$$

Show for this derived game v_m it holds that its Harsanyi dividends are determined by

$$\Delta_{v_m}(S) = |S| \cdot \Delta_v(S)$$

and that indeed $m(N, v) = \varphi(N, v_m)$, where φ is the Shapley value.

Problem 4.8 Define the potential function $Q\colon \mathcal{G} \to \mathbb{R}$ by $Q(\varnothing, v) = 0$ and for every (v, N) with $N \neq \varnothing$:

$$Q(N, v) = \tfrac{v(N)}{|N|}.$$

With this potential Q we introduce the corresponding allocation rule $\xi : \mathcal{G}^N \to \mathbb{R}^N$ with

$$\xi_i(N, v) = Q(N, v) - Q(N - i, v) = \tfrac{v(N)}{|N|} - \tfrac{v(N-i)}{|N|-1}$$

for all players $i \in N$.

(a) Show that this indeed introduces a proper allocation rule on $\mathcal{G}$ that satisfies the balanced payoff property.
(b) Show that

$$v_\xi(S) = \sum_{i \in S} \xi_i(S, v) = v(S) - \sum_{T \subsetneq S} \tfrac{|S|-|T|}{|S|-1} \Delta_v(T),$$

where Δ_v denotes the Harsanyi dividends of the game v.
(c) Show that $\xi(N, v) = \varphi(N, v_\xi)$.

Problem 4.9 (*The Proportional Value*) The Proportional Value has been introduced by Ortmann (2000) as an analogue to the Shapley value for a proportional variation of the cooperative potential concept. In this problem I discuss and develop Ortmann's alternative approach. Following the definition of a value-based potential, Ortmann introduced a ratio-based reformulation: A positive allocation rule[7] ϕ on G admits a *ratio-potential* if there exists a strictly positive function $P\colon \mathcal{G} \to \mathbb{R}_{++}$ such that for every $i \in N$

$$\phi_i(N, v) = \frac{P(N, v)}{P(N - i, v)}.$$

Furthermore, we can now formulate the analogue of the balanced payoff property for ratios. Indeed, a value ϕ is said to *preserve ratios* if for all games $(N, v) \in \mathcal{G}$ and players $i, j \in N$ with $i \neq j$:

$$\frac{\phi_i(N, v)}{\phi_i(N - j, v)} = \frac{\phi_j(N, v)}{\phi_j(N - i, v)}.$$

Finally, we define the analogue of the property that a value is standard for two player games. In this context, a value ϕ on $\mathcal{G}$ is *proportional for two player games* if for all $(N, v) \in \mathcal{G}$ with $N = \{i, j\} = ij$ it holds that

$$\phi_i(ij, v) = v(i) + \frac{v(i)}{v(i) + v(j)} \cdot \left[\, v(ij) - v(i) - v(j) \,\right] \equiv \frac{v(i)}{v(i) + v(j)} \cdot v(ij).$$

Solve the following problems for this framework.

(a) Show that a value ϕ admits a ratio-potential if and only if ϕ preserves ratios.
(b) Show that there exists exactly one value ψ on G that is efficient and preserves ratios. This value is called the *Proportional value* and is given recursively by $\psi_i(\{i\}, v) = v(\{i\})$ and

$$\psi_i(N, v) = \frac{v(N)}{1 + \sum_{j \in N - i} \frac{\psi_j(N - i, v)}{\psi_i(N - j, v)}} \tag{4.43}$$

(c) Show that the Proportional value ψ is proportional for two player games and satisfies HM-consistency.
(d) Show that the Proportional value ψ is the unique allocation rule on G that satisfies proportionality for two player games as well as HM-consistency.

Problem 4.10 Recall that a share function λ on G satisfies the nullifying player property if for every nullifying player i in (N, v)[8] it holds that $\lambda_i(N, v) = 0$.

[7] A value or allocation rule is "positive" if it only assigns strictly positive values $\phi(N, v) \gg 0$ to the players in the cooperative game (N, v).

[8] Recall that a nullifying player in the game (N, v) is a player $i \in N$ such that $v(S) = 0$ whenever $i \in S$.

(a) Show that the DP share function is the unique share function that satisfies efficient sharing, symmetry, μ^d-linearity and the nullifying player property, where $\mu^d(N, v) = \sum_{S \subset N} v(S)$.
(b) Derive a closed expression of the unique BL-share function λ^{μ^d} for μ^d.

Problem 4.11 Consider the function $\mu^B : \mathcal{G} \to \mathbb{R}_+$ given by

$$\mu^B(N, v) = \frac{1}{2^{|N|-1}} \sum_{S \subset N} (\, 2|S| - |N| \,)\; v(S).$$

Give a complete characterization of the BL-share function $\beta = \lambda^{\mu^B}$ that corresponds to this function μ^B. Also determine the corresponding share potential.[9]

[9] This BL-share function is known as the Banzhaf share function, which is the normalization of the so-called Banzhaf value. The Banzhaf value was seminally introduced in Banzhaf (1965). Dragan (1996) subsequently investigated the corresponding Banzhaf potential. The normalized Banzhaf value and the corresponding share function have been fully characterized in van den Brink and van der Laan (1998b).

Chapter 5
Directed Communication Networks

In this and the next chapter I discuss some applications of the cooperative game theoretic notions that were covered in the previous chapters. In this chapter I focus on the theory of directed networks. In general, a *network* is a list of binary relationships or *links* between pairs of individuals. There are multiple ways in which we can interpret such binary relationships. Here I address first the various interpretations of what a link represents.

A network is *undirected* if each relationship is fully equal in the sense that the two parties interacting do so on a completely equal basis. In the economy and social life there are numerous situations that can be represented by such undirected networks. The main economic application is that of trade between fully informed corporations in a market setting. These networks are usually denoted as *trade networks*. In social life, undirected networks can be used to represent communication between equal parties such as social conversations and gatherings. Here, I explicitly assume that in such conversations the information transfer is either fully mutual or inconsequential.[1] Such networks are usually denoted as communication networks.

A network is *directed* if each relationship has a directed nature in the sense that within the relationship there is some principal individual or *initiator* from whom the relationship emanates and a receiving agent or *respondent* at whom the relationship is directed. Hence, within each directed relationship the two constituting individuals do not participate on equal terms and each relationship or link goes from one individual (the initiator) to the other (the respondent).

These directed networks express a range of social structures, events, and situations. I limit myself to four obvious interpretations of such directed networks that result in rather different formal representations and interpretations.

Information networks First, directed networks can be used to describe particular communication situations in which the transfer of valuable information is represented. The individuals in the network are persons that communicate

[1] The information transfer is mutual if the value of the information is equal; the traded information thus cancels in economic terms. Inconsequential information transfer means that the information that is communicated has no value to the recipient.

R.P. Gilles, *The Cooperative Game Theory of Networks and Hierarchies*, Theory and Decision Library C 44, DOI 10.1007/978-3-642-05282-8_5,

with one another. In a directed communication situation within any communication link one person is communicating to another and the other is only listening. The information transferred communicated from the initiator to the respondent is assumed to have value for the respondent. An excellent example of such a network is that of gossiping between persons in a social community. In such situations, one person is transferring information to another and the transferred information is of social value to the recipient. So, there is a one-way flow of information from one person to the other. This can be denoted as an *information (transfer) network*.

Domination networks Second, directed networks can be used to describe sports competitions of all sorts. One main application here is the representation of sports tournaments. A victory of one sports team over another is represented by a directed relationship. Theory has been developed to measure how well various teams perform in such sports competitions. The higher the measure the better the performance of the team in consideration. In the rest of the chapter I refer to this as the measurement of *dominance*.

Authority networks The third main application concerns the description of authority situations. Here a link describes an unambiguous authority relationship between two individuals. One individual (the "principal") exercises some form of authority over the other individual (the "agent"). The relationships in such authority networks can also be denoted as principal-agent relationships. In economics this representation is used to describe authority situations in production organizations such as firms. Usually the authority relationship between the principal and agent is complicated by the fact that the agent possesses valuable, important information to execute the task at hand to which the principal does not have access. Which incentives have to be used to induce the agent to act fully on behalf of the principal in such circumstances? This question has resulted into so-called "principal-agent" theory, which is a major sub-field in non-cooperative game theory. For a comprehensive source on relevant contributions to the theory of employment relations I refer to Putterman (1986) and for a thorough discussion of a mathematical, cooperative game theoretic approach I refer to Ichiishi (1993).

Permission structures An important subclass of authority networks are those that explicitly represent veto power situations. Within these directed networks, the authority relationship between the initiator and respondent is further strengthened. An initiator is now assigned the power to fully control the respondent and is now denoted as a "superior", while the respondent is interpreted as a "subordinate". The main interpretation is that in the context of a production process, which technology is proprietary. Here, a superior can essentially deny a subordinate the full access to the production technology, thus controlling the subordinate in creating any added value. This introduces a form of veto power of the superior over the subordinate regarding the access of the subordinate to the production technology. Directed networks representing such veto situations are known as permission structures in the literature. A full development of this field is given in Chapter 6.

I emphasize here that the various interpretations of directed networks lead to very different analytical concepts and questions. In this chapter I discuss concepts that are used to analyze information and domination networks. In Chapter 6 I develop a cooperative game theoretic perspective on permission structures.

In this chapter I first discuss the theory that measures the dominance position of individuals within direct networks, more specifically domination networks. This theory provides tools for measuring power in authority situations as well as performance in sports competitions. This subject of dominance measurement is closely related to the measurement of "centrality" in such social networks (Wasserman and Faust, 1994). The measurement of centrality has a long standing tradition in the social sciences, in particular the theory of social networks.

Next I discuss how cooperative games are affected if its players are structured according to some directed network. Here a directed link in the network is interpreted explicitly as an information transfer relation or a weak authority relationship founded on an informational asymmetry. Cooperative games can be used to describe values generated in such information networks, especially the outputs resulting from the use of shared information.

5.1 Directed Networks

At the onset of the technical discussion in this chapter I first consider the foundational notions of the theoretical framework, namely the concept of a directed network. I will use some non-standard notation in the rest of this section.

Let $N = \{1, \dots, n\}$ be a finite set of players. A player is here explicitly interpreted as an individual within a directed network representing a certain dominance structure. As described above, a *directed network* on N is a collection of dominance relations which can be represented by a mapping $D \colon N \to 2^N$ such that $i \notin D(i)$ for every individual $i \in N$. Here the set $D(i) \subset N$ denotes the set of individuals $j \in D(i)$ that succeed $i \in N$ in D. Hence, if $j \in D(i)$, then individual j is a *successor* to player i.[2]

An alternative notation employed in the literature on directed networks is that a directed network is represented by $D \subset N \times N$ such that $(i, i) \notin D$ for every $i \in N$. Now $(i, j) \in D$ is interpreted to mean that player i precedes player j, i.e., player j succeeds player i and in that regard player j is a successor of i. In graph theory the pair (i, j) is also known as an *arc* in the directed network D.

The collection of all directed networks on the player set N is denoted by $\mathcal{D}^N$.

Let $D \in \mathcal{D}^N$ be a directed network on N. The directed network D can also be represented by the inverse of the mapping D given by $D^{-1} \colon N \to 2^N$ with for every $i \in N$:

---

[2] Here the notion of succession should be viewed rather widely. It could mean so little as being the recipient of a piece of information from another person, or being the loser in a sports match with the other player, or so much as being a ward of a person in a position of authority in the capacity of being her guardian.

$$D^{-1}(i) = \{j \in N \mid i \in D(j)\}. \tag{5.1}$$

The relationship $j \in D^{-1}(i)$ can be interpreted as that player j precedes player i in D. Hence, player j is a *predecessor* of player i in the directed network D. It should be clear that the mapping D^{-1} contains the same information as the original mapping D.

A (directed) *path* from player i to player j in D is a finite sequence of players $P = (i_1, \ldots, i_m)$ such that $i_1 = i$, $i_m = j$ and $i_{k+1} \in D(i_k)$ for all $k \in \{1, \ldots, m-1\}$. A path $P = (i_1, \ldots, i_m)$ from a player i to herself, i.e., $i_1 = i_m = i$, is also denoted as a (directed) *cycle*. A directed network D is called *acyclic* if it does not contain any cycles.

The notion of a path allows an alternative approach to directed networks. The mapping $D^+ : N \to 2^N$ is the *transitive closure* of the directed network D if for every player $i \in N$:

$$D^+(i) = \{j \in N \mid \text{There exists path from } i \text{ to } j \text{ in } D\}. \tag{5.2}$$

The relationship $j \in D^+(i)$ can be interpreted as that player j is the "indirect" or "remote" successor of player i. Note that there exists a cycle in the directed network D if there exists some player $i \in N$ with $i \in D^+(i)$. That is, the directed network D is acyclic if and only if $i \notin D^+(i)$ for every $i \in N$.

Similarly, the inverse of the transitive closure D^+ of D can be given by the mapping $D^- : N \to 2^N$ with

$$D^-(i) = \{j \in N \mid \text{There exists path from } j \text{ to } i \text{ in } D\}. \tag{5.3}$$

Here, $j \in D^-(i)$ can be called a remote or indirect predecessor of player i in the directed network D.

Example 5.1 Consider a simple example of a directed network on the player set $N = \{1, 2, 3, 4\}$. Let $D \in \mathcal{D}^N$ be given by $D(1) = \varnothing$, $D(2) = \{1, 3\}$, $D(3) = \{4\}$, and $D(4) = \{2\}$. The directed network D is graphically represented in Fig. 5.1. It is clear from the definition of D and its graphical representation that D contains a cycle, namely $(2, 3, 4)$.

The inverse mapping is now determined by $D^{-1}(1) = \{2\}$, $D^{-1}(2) = \{4\}$, $D^{-1}(3) = \{2\}$ and $D^{-1}(4) = \{3\}$. Hence, although player 1 has no successors, all players have predecessors in this network.

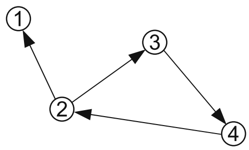

Fig. 5.1 A directed communication network

Alternatively the directed network D as depicted can be represented by its transitive closure: $D^+(1) = \varnothing$ and $D^+(2) = D^+(3) = D^+(4) = N$ and its inverse, $D^-(1) = D^-(2) = D^-(3) = D^-(4) = \{2,3,4\}$. ■

A directed network D is a *hierarchy* if D is acyclic and there exists a unique player $i_0 \in N$ with $D^{-1}(i_0) = \varnothing$. Hierarchies are used to describe (formal) authority relationships between members of a production organization. The relationships can therefore be reinterpreted in this context of the exercise of authority. In that regard player $j \in D(i)$ is a subordinate rather than a mere successor. And a player $j \in D^{-1}(i)$ is a superior rather than a predecessor. In a hierarchy the player i_0 is the global superior and all other players are her indirect subordinates. In Chapter 6, player i_0 is denoted as the "executive" in the hierarchy D.

Finally, a directed network $T \in \mathcal{D}^N$ is a *subnetwork* of the directed network $D \in \mathcal{D}^N$ if $T(i) \subset D(i)$ for every player $i \in N$.

5.2 Measuring Dominance in Directed Networks

In this section I first ask how we can measure the overall "dominance"—or, alternatively, the connectedness—of an individual player in the network D. As stated before, this is a crucial question in the evaluation of the performance of these individuals when the network represents a communication situation or a sports competition.

In this section I introduce and discuss three different measures. The first one is the simplest of these measures and is based on the so-called *degree* of a player in the network. The degree is the number of links that a player has with other individuals in the network. This introduces the degree measure. The degree measure has a widespread use in the evaluation of sports competitions. Indeed, in a sports tournament the degree measures simply is equal to the number of matches that is won by a particular player.

Second, I introduce a very different and more subtle measure to express the dominance power of a certain player in a directed network. The β-measure was introduced by van den Brink and Gilles (1994) and subsequently analyzed thoroughly in van den Brink and Gilles (2000). It measures the performance of a player in a weighted fashion. Dominance over a player that is dominated by many other players is weighted less than the dominance over a player that is only dominated by a few other players. The β-measure thus incorporates an evaluation of the various links in the network.[3] This β-measure turns out to have many appealing properties, which are explored below. Some variations on the β-measure are discussed as well.

Finally, I discuss a class of iterated measures in directed networks. These iterated measures are based on the repeated application of a certain dominance measure on

[3] A similar philosophy is incorporated in the co-author model of Jackson and Wolinsky (1996) in which the value of a well-involved co-author is valued less than the cooperation with a co-author who still has significant time to devote to a joint project since he is less involved with other co-authors.

the same network. This process might result into a converging sequence, which limit is investigated as a possible dominance measure in itself.

5.2.1 The Degree Measure

Formally, a *measure* is a function $m\colon \mathcal{D}^N \to \mathbb{R}^N$ that assigns to every player $i \in N$ in some directed network $D \in \mathcal{D}^N$ a number $m_i(D)$ that measures the centrality or dominance of that player in the situation represented by the directed network D. Usually one limits oneself to measures that assign non-negative evaluations, i.e., usually it is assumed that $m_i(D) \geqq 0$.

The degree measure has a long standing tradition in the use of directed networks, in particular for the representation of sports competitions. It simply measures the dominance of a certain player by the number of wins that player has scored. Therefore, the degree measure is also known as the "score measure".

Definition 5.2 The *degree measure* is the mapping $\delta\colon \mathcal{D}^N \to \mathbb{R}^N_+$ which is defined by $\delta_i(D) = |D(i)|$ for every directed network $D \in \mathcal{D}^N$ and every player $i \in N$.

In this section I first discuss an axiomatization of the degree measure. This axiomatization has been developed in van den Brink and Gilles (2000) and is based on some simple formulations of some straightforward properties.

Before I can formulate these properties, I have to introduce some auxiliary notation. A *partition* of the directed network $D \in \mathcal{D}^N$ is a collection of directed networks $\{D_1, \ldots, D_m\} \subset \mathcal{D}^N$ such that for every player $i \in N$:

$$D(i) = \bigcup_{k=1}^{m} D_k(i) \tag{5.4}$$

$$D_k(i) \cap D_h(i) = \varnothing \quad \text{for all } k \neq h. \tag{5.5}$$

A partition $\{D_1, \ldots, D_m\} \subset \mathcal{D}^N$ of D is called *independent* if each player is dominated in at most one sub-network in this partition, i.e., if for all players $i \in N$ it holds that

$$\#\{k \mid k \in \{1, \ldots, m\} \text{ and } D_k^{-1}(i) \neq \varnothing\} \leqq 1. \tag{5.6}$$

In an independent partition the players are partitioned with regard by whom they are dominated. Indeed, in such a partition each player is partitioned with all her predecessors into exactly one of the sub-networks. This independence property turns out to be crucial for our main characterization of the degree measure.

Axiom 5.3 *Let $m\colon \mathcal{D}^N \to \mathbb{R}^N$ be some measure on the collection of all directed networks. We introduce the following properties for the measure m.*

Degree normalization *For every directed network $D \in \mathcal{D}^N$ it holds that*

$$\sum_{i \in N} m_i(D) = |D|. \tag{5.7}$$

Dummy position property *For every directed network $D \in \mathcal{D}^N$ and every player $i \in N$ with $D(i) = \varnothing$ it holds that $m_i(D) = 0$.*[4]

Symmetry *For every directed network $D \in \mathcal{D}^N$ and all players $i, j \in N$ with $D(i) = D(j)$ and $D^{-1}(i) = D^{-1}(j)$ it holds that $m_i(D) = m_j(D)$.*

Additivity over independent partitions *For every directed network $D \in \mathcal{D}^N$ and every independent partition $\{D_1, \ldots, D_m\} \subset \mathcal{D}^N$ of D it holds that*

$$m(D) = \sum_{k=1}^{m} m(D_k). \tag{5.8}$$

The properties formulated in Axiom 5.3 exactly characterize the degree measure on the collection of all directed networks. The proof of the next characterization is relegated to the appendix of this chapter.

Theorem 5.4 (van den Brink and Gilles, 2000, Theorem 3.3) *A measure m on the collection of all directed networks $\mathcal{D}^N$ is equal to the degree measure δ if and only if m satisfies all properties stated in Axiom 5.3.*

In order to check that the characterization given in Theorem 5.4 is proper, the independence of the four stated axioms has to be confirmed. For a measure that satisfies all properties except degree normalization I refer to the β-measure that is discussed in the next subsection. For the other three properties, I consider three other measures on $\mathcal{D}^N$ introduced in van den Brink and Gilles (2000).

(i) Consider the egalitarian measure m^1 on $\mathcal{D}^N$, which for every $i \in N$ is given by

$$m_i^1(D) = \frac{|D|}{|N|} \tag{5.9}$$

This measure satisfies obviously degree normalization, symmetry and additivity over independent partitions. Obviously, m^1 does not satisfy the dummy position property since all participating players attain an equal measure under m^1.

(ii) Next let m^2 be a measure on $\mathcal{D}^N$, which for every $i \in N$ is given by

$$m_i^2(D) = \sum_{j \in D(i)\colon\, i = \min\{h \mid h \in D^{-1}(j)\}} \#D^{-1}(j) \tag{5.10}$$

[4] A player without any successors in a network D is also said to "occupy a dummy position" in D. This explains the name of this axiom.

This measure considers the ranking of players according to the in-degree of those successors for which they are the "first" predecessor. Here the listing of players $1, \ldots, n$ is used as a precedence order.
The measure m^2 satisfies degree normalization, the dummy position property as well as additivity over independent partitions. However, it obviously is not symmetric due to the usage of the precedence order over the players.

(iii) Finally, consider the measure m^3 on $\mathcal{D}^N$, which for every $i \in N$ is given by

$$m_i^3(D) = \frac{|D|}{\#\{j \in N \mid D^{-1}(j) \neq \varnothing\}} \sum_{j \in D(i)} \frac{1}{\#D^{-1}(j)} \tag{5.11}$$

This measure is in fact a re-normalization of the β-measure as will be discussed below. The measure m^3 satisfies degree normalization, the dummy position property and symmetry. However, m^3 is not additive over independent partitions. Indeed, consider $N = \{1, 2, 3, 4\}$ and D given by $D(1) = \{2, 3\}$, $D(2) = D(3) = \{4\}$ and $D(4) = \varnothing$. An independent partition of D is now given by $\{D_1, D_2\}$ with $D_1(1) = \{2, 3\}$ and $D_1(2) = D_1(3) = D_1(4) = \varnothing$ and with $D_2(2) = D_2(3) = \{4\}$ and $D_2(1) = D_2(4) = \varnothing$.
We now can compute that $m^3(D) = \frac{2}{3}(4, 1, 1, 0)$, $m^3(D_1) = (2, 0, 0, 0)$ and $m^3(D_2) = (0, 1, 1, 0)$. This confirms that $m^3(D) \neq m^3(D_1) + m^3(D_2)$.

The three directed networks used for measure m^3—discussed in (iii) above—are depicted in Figs. 5.2, 5.3 and 5.4. These graphical representations also illustrate the concept of an independent partition of a directed network into a set of sub-networks.

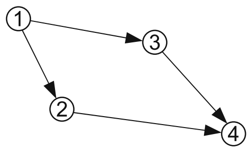

Fig. 5.2 The directed communication network D

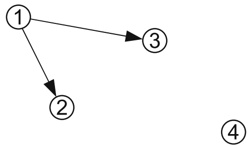

Fig. 5.3 The directed communication network D_1

These dominance or "power" measures are mostly used to rank the participating players in a directed network. In particular, when the network represents the results in some sports competition, these measures have the purpose to rank the participants in these competitions.

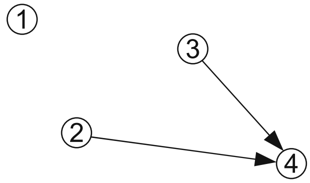

Fig. 5.4 The directed communication network D_2

The ranking based on the degree measure is particularly important in these applications. On the class of so-called tournaments,[5] the ranking method based on the degree measure has been axiomatized by Rubinstein (1980). This restricted axiomatization has been extended in a natural fashion to the whole space of directed networks $\mathcal{D}^N$ by van den Brink and Gilles (2003). It turns out that this particular ranking method is based on some plausible properties.

It would go too far to discuss these ranking methods and axiomatizations in the context of these lecture notes. It would deviate too much from the objectives of these notes. Therefore, I refer to these papers for the details.

5.2.2 The β-Measure

The main measure introduced by van den Brink and Gilles (2000) incorporates the weight of a directed relationship. Indeed, in the case of a sports contest—or, more generally, any tournament—the value of one team's victory over another team is diminishing with the number of defeats that the conquered team has suffered. The most direct expression of this is to assume that the value of such a victory is the inverse of the number of defeats suffered by the conquered team. This idea underlies the β-measure.

Formally, the β-*measure* is the function $\beta \colon \mathcal{D}^N \to \mathbb{R}^N$ defined by

$$\beta_i(D) = \sum_{j \in D(i)} \frac{1}{\#D^{-1}(j)} \quad \text{for every } i \in N. \tag{5.12}$$

To give some computational comparisons, consider the directed network D depicted in Figure 5.2. For this network I compute the degree measure as $\delta(D) = (2, 1, 1, 0)$ and the β-measure as $\beta(D) = (2, \frac{1}{2}, \frac{1}{2}, 0)$. It is clear that under the β-measure the difference between player 1 and players 2 and 3 is larger than under the degree measure. This is mainly due to the idea that under the β-measure dominance over player 4 by players 2 and 3 is simply worth less than the dominance of player 1 over players 2 and 3.

[5] A directed network $D \in \mathcal{D}^N$ is a *tournament* if it holds for all pairs $i, j \in N$ that either $i \in D(j)$ or $j \in D(i)$.

On the other hand, in the directed network D' depicted in Fig. 5.1 the main measures are computed as $\delta(D') = \beta(D') = (0, 2, 1, 1)$. Hence, there is no difference between the two measures. This is due to the special circumstance that all teams are beaten exactly once.

The surprising fact about the β-measure is that it satisfies all properties that the degree measure satisfies except for the normalization. The degree measure satisfies the degree normalization properties, while the β-measure is normalized by the number of dominated players within the network.

Definition 5.5 A measure $m \colon \mathcal{D}^N \to \mathbb{R}^N$ satisfies *dominance normalization* if for every directed network $D \in \mathcal{D}^N$ it holds that

$$\sum_{i \in N} m_i(D) = \#\{j \in N \mid D^{-1}(j) \neq \varnothing\}. \tag{5.13}$$

The next theorem states the main characterization of the β-measure, using the same axioms as used in the characterization of the degree measure stated in Theorem 5.4.

Theorem 5.6 *A measure m on $\mathcal{D}^N$ is equal to the β-measure if and only if m satisfies dominance normalization, the dummy position property, symmetry and additivity over independent partitions.*

To show the independence of the four axioms in Theorem 5.6 I introduce four examples. Each example discusses a measure that satisfies three out of the four axioms listed.

(i) Consider the degree measure δ. Note that this measure satisfies all axioms except for dominance normalization.

(ii) Consider the measure m^4 on $\mathcal{D}$ defined by

$$m_i^4(D) = \frac{\#\{j \in N \mid D^{-1}(j) \neq \varnothing\}}{|N|}. \tag{5.14}$$

This measure satisfies dominance normalization, symmetry and additivity over independent partitions. However, m^4 does not satisfy the dummy position property. This is immediately clear from the property that every player gets an equal share of the total points awarded, being the number of dominated players in D.

(iii) Let the players in N be labelled $1, \ldots, n$. Now let the measure m^5 be given by

$$m_i^5(D) = \#\{j \in D(i) \mid i = \min\{h \mid h \in D^{-1}(j)\}\}. \tag{5.15}$$

This measure satisfies dominance normalization, the dummy position property and additivity over independent partitions. On the other hand, m^5 does not satisfy symmetry. For the directed network D given in Fig. 5.2, we compute $m^5(D) = (2, 1, 0, 0)$, which shows that the equipoised players 2 and 3 are treated differently.

(iv) Finally, let m^6 be a measure on $\mathcal{D}$ given by

$$m_i^6(D) = \frac{\#\{j \in N \mid D^{-1}(j) \neq \varnothing\}}{|D|} \cdot \delta_i(D). \tag{5.16}$$

The measure m^6 is a re-normalization of the degree measure δ. This measure satisfies dominance normalization, the dummy position property as well as symmetry. However, m^6 does not satisfy additivity over independent partitions. Consider the partition of D depicted in Figs. 5.2, 5.3 and 5.4. Then $m^6(D) = (1\frac{1}{2}, \frac{3}{4}, \frac{3}{4}, 0)$, $m^6(D_1) = (2, 0, 0, 0)$ and $m^6(D_2) = (0, \frac{1}{2}, \frac{1}{2}, 0)$. Thus, $m^6(D) \neq (2, \frac{1}{2}, \frac{1}{2}, 0) = m^6(D_1) + m^6(D_2)$.

The β-measure is an interesting departure from the standard measurement of performance or dominance in directed networks. Indeed, one usually resorts to the degree measure to measure such performance. Although the degree measure has some interesting properties, the β-measure is much more closely in tune with the cooperative game theoretic tools such as the Shapley value. This is analyzed next.

5.2.2.1 The Successor Games

I first consider the cooperative game theoretic foundation of the β-measure. With each directed network $D \in \mathcal{D}^N$ two corresponding cooperative "successor games" can be introduced.

Definition 5.7 Let $D \in \mathcal{D}^N$ and define for every coalition of players $S \subset N$

$$\bar{D}(S) = \bigcup_{i \in S} D(i) \tag{5.17}$$

as the *successor coalition* of S in the sense that this is the coalition of the players that are successors in D to players in S.

(a) With the directed network D now define the cooperative game $p_D \in \mathcal{G}^N$ as the *optimistic successor game* corresponding to the directed network D by

$$p_D(S) = \#\bar{D}(S) \quad \text{for every coalition } S \subset N. \tag{5.18}$$

(b) Similarly, for the network D define the cooperative game $s_D \in \mathcal{G}^N$ as the *conservative successor game* corresponding to the directed network D by

$$s_D(S) = \#\{j \in N \mid D^{-1}(j) \subset S\} \quad \text{for every coalition } S \subset N. \tag{5.19}$$

The optimistic successor game has been introduced in van den Brink and Gilles (2000), while the conservative successor game is the subject of van den Brink and Borm (2002). The optimistic successor game p_D has a very simple structure, but is still rather complex from a cooperative game theoretic point of view. Indeed, the optimistic successor game assigns to every coalition those players that have at least one predecessor in that coalition. Thus, each coalition also assumes to have control of those players that are partially dominated by players in that coalition.

On the other hand, the conservative successor game s_D only counts those players that are completely dominated by the players in the given coalition. In this regard the game s_D is indeed more conservative in the assignment of value as the optimistic successor game p_D. Furthermore, s_D has a very beautiful cooperative game theoretic structure and in that regard has superior cooperative game theoretic properties than p_D. The next lemma is straightforward and a proof is therefore omitted. For details I also refer to van den Brink and Borm (2002).

Lemma 5.8 *Let $D \in \mathcal{D}^N$ be some directed network. Then it holds that*

$$s_D = \sum_{i \in \bar{D}(N)} u_{D^{-1}(i)} \tag{5.20}$$

where u_S denotes the unanimity basis game corresponding to the coalition $S \subset N$.

From the stated property, one immediately concludes that the conservative successor game is a convex game and that its Harsanyi dividends are very simple and non-negative. The assertion of the lemma is also illustrated by the next example.

Example 5.9 Consider the directed network D on $N = \{1, 2, 3, 4\}$ depicted in Fig. 5.5. For D we have that $\beta(D) = (0, 1\frac{1}{2}, 1\frac{1}{2}, 0)$.

With the network D one can now compute the two corresponding successor games. In these two successor games the players 1 and 4 are clearly two dummy players. The next table summarizes the main characteristics of these games under exclusion of players 1 and 4:

From the computed Harsanyi dividends it is clear that $p_D = 2u_2 + 2u_3 - u_{23}$ and $s_D = u_2 + u_3 + u_{23}$.

S	$p_D(S)$	$\Delta_{p_D}(S)$	$s_D(S)$	$\Delta_{s_D}(S)$
2	2	2	1	1
3	2	2	1	1
23	3	−1	3	1

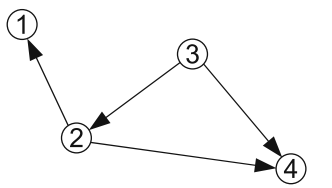

Fig. 5.5 The directed communication network D in Example 5.9

Thus, we compute straightforwardly that $\varphi(p_D) = (0, 1\frac{1}{2}, 1\frac{1}{2}, 0)$ and $\varphi(s_D) = (0, 1\frac{1}{2}, 1\frac{1}{2}, 0)$. Hence, we arrive at the remarkable property that $\beta(D) = \varphi(p_D) = \varphi(s_D)$. This is no coincidence as we show next. ■

The following theorem has seminally been developed by van den Brink and Gilles (2000) for the optimistic successor game and extended by van den Brink and Borm (2002) for the conservative successor game. The assertions link the β-measure with its game theoretic foundation as the Shapley value of the corresponding successor game.

Theorem 5.10 (β-measure equivalence) *Let $D \in \mathcal{D}^N$ be some directed network. Then the β-measure of D is equal to the Shapley value of the two corresponding successor games, i.e.,*

$$\beta(D) = \varphi(p_D) = \varphi(s_D). \tag{5.21}$$

From the assertion stated in Theorem 5.10 there are two other conclusions to be drawn regarding the conservative successor game and the β-measure. First, I introduce some auxiliary concepts and notation.

Let $D \in \mathcal{D}^N$ be some directed network on N. A directed network $T \in \mathcal{D}^N$ is a *simple subnetwork* of D if $T(i) \subset D(i)$ for all $i \in N$ and $\#T^{-1}(i) = 1$ for every dominated player $i \in \bar{D}(N) = \{j \in N \mid D^{-1}(j) \neq \varnothing\}$. The collection of all simple subnetworks of D is denoted by $\Sigma(D)$.

Using simple subnetworks of a directed network we can reformulate the β-measure as an average of degree measures of these simple subnetworks. The next corollary was originally stated in van den Brink and Borm (2002).

Corollary 5.11 *For every directed network $D \in \mathcal{D}^N$ it holds that*

$$\beta(D) = \frac{1}{\#\Sigma(D)} \sum_{T \in \Sigma(D)} \delta(T) \tag{5.22}$$

Proof To show the assertion I use the definition of the β-measure directly.

First, note that $\#\Sigma(D) = \prod_{h \in \bar{D}(N)} \#D^{-1}(h)$. Therefore, for all $i, j \in N$ with $j \in D(i)$:

$$\#\{T \in \Sigma(D) \mid j \in D(i)\} = \prod_{h \in \bar{D}(N) \setminus \{j\}} \#D^{-1}(h).$$

Thus, for every $i \in N$

$$\beta_i(D) = \sum_{j \in D(i)} \frac{1}{\#D^{-1}(j)} = \frac{\sum_{j \in D(i)} \left(\prod_{h \in \bar{D}(N) \setminus \{j\}} \#D^{-1}(h) \right)}{\prod_{h \in \bar{D}(N)} \#D^{-1}(h)}$$
$$= \frac{1}{\#\Sigma(D)} \sum_{j \in D(i)} \#\{T \in \Sigma(D) \mid j \in T(i)\}$$
$$= \frac{1}{\#\Sigma(D)} \sum_{T \in \Sigma(D)} \#T(i) = \frac{1}{\#\Sigma(D)} \sum_{T \in \Sigma(D)} \delta_i(T)$$

This shows the assertion. ■

Second, I recall that a convex cooperative game has the property that its Core is exactly the convex hull of the marginal contribution vectors of that game. In their analysis, van den Brink and Borm (2002) used the convexity of the conservative successor game to derive a characterization of its Core. For a proof of the next corollary I refer to that paper.

Corollary 5.12 *For every directed network $D \in \mathcal{D}^N$ it holds that*

$$C(s_D) = \text{Conv}\,\{\, \delta(T) \mid T \in \Sigma(D) \,\}. \tag{5.23}$$

5.2.2.2 The Modified β-Measure

It can be assumed that each individual in a network "succeeds herself" once. Hence, artificially a link from every player i to herself is added to the directed network representations. This does not alter the degree measure since every player artificially is assigned an additional point. However, the β-measure changes significantly. I can reformulate the β-measure for the directed network with this addition as

$$\beta_i'(D) = \sum_{j \in D(i) \cup \{i\}} \frac{1}{\#D^{-1}(j) + 1} \tag{5.24}$$

This measure β' is known as the *modified* β-measure and has been seminally introduced by van den Brink and Gilles (1994). Subsequently, Borm, van den Brink, and Slikker (2002) used the modified β-measure to discuss the iterated application of a measure to some directed network.[6]

Following Borm et al. (2002) we introduce for every directed network $D \in \mathcal{D}^N$ on N the *modified conservative successor game* $s_D' \in \mathcal{G}^N$ by

$$s_D'(S) = \#\,\{j \in S \mid D^{-1}(j) \subset S\} \quad \text{for every coalition } S \subset N. \tag{5.25}$$

[6] The iterated application of measures is discussed extensively in the next section of this chapter. There it is also argued that the modified β-measure rather than the standard β-measure is especially suited for iterated application.

Hence, the modified conservative successor game assigns to every coalition the number of its members whose predecessors are all members of that coalition as well. Hence, membership of the coalition is required now. Without proof I now state a slight modification of the main result reported above.

Lemma 5.13 *For every directed network $D \in \mathcal{D}^N$ it holds that its modified β-measure is equal to the Shapley value of the modified conservative successor game, i.e., $\beta'(D) = \varphi(s'_D)$.*

Also remark that the modified β-measure does not satisfy dominance normalization, but rather node normalization: $\sum_{i \in N} \beta'_i(D) = |N|$. The reason is that in this modified approach each player succeeds herself. This normalization makes the modified β-measure particularly interesting to use to measure performance of teams in sports competitions. This assessment is confirmed by a simple application of the previously discussed measures to a sports competition or "tournament".

5.2.2.3 An Illustration

Consider a soccer competition between four teams. They play a complete round-robin and the results are as follows: Team 1 beats Team 3 and draws with Team 4; Team 2 beats Teams 1 and 4; Team 3 beats Teams 2 and 4; and Team 4 just draws with Team 1. If international scoring rules are applied, then a win gives 3 points, a draw 1 point, and a loss 0 points. Following this scoring rule we get a performance measure given by $S = (4, 6, 6, 1)$.

This sports competition can be represented by two different directed networks due to the treatment of the drawn game between Teams 1 and 4. The first representation D_1 given in Fig. 5.6 disregards the draw between Teams 1 and 4 and only represents the wins by directed links in the network. In this representation the degree measure is computed as $\delta(D_1) = (1, 2, 2, 0)$ and the β-measure computed as $\beta(D_1) = (1, 1\frac{1}{2}, 1\frac{1}{2}, 0)$. So, exactly the same ranking is generated by international rules as by the degree and β-measures for this representation.

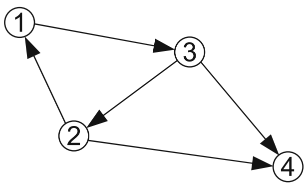

Fig. 5.6 Representation D_1 of the sports competition

There is an alternative representation of this sports competition, denoted by D_2. Here the draw of Teams 1 and 4 is represented by two directed links between Teams 1 and 4, indicated by the outside arrows in Fig. 5.7. In this modified representation I compute the degree measure as $\delta(D_2) = (2, 2, 2, 1)$ and the β-measure by $\beta(D_2) = (1\frac{1}{3}, \frac{5}{6}, 1\frac{1}{3}, \frac{1}{2})$. These computations make clear that the degree measure performs

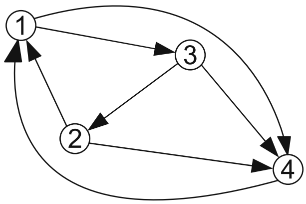

Fig. 5.7 Representation D_2 of the sports competition

much worse in this modified representation. However, the β-measure improves in its performance as an evaluator of this sports competition.

There is yet another modification possible in these network representations of this sports competition. It can be assumed that each team "beats itself" once in this competition. Hence, artificially a link from every team i to itself is added to these network representations. This exactly corresponds to the application of the modified β-measure to this sports competition. For the two given representations I now compute $\beta'(D_1) = (1, 1\frac{1}{3}, 1\frac{1}{3}, \frac{1}{3})$ and $\beta'(D_2) = (1\frac{1}{12}, 1\frac{1}{12}, 1\frac{1}{4}, \frac{7}{12})$. This changes the assessment given by the computed β-measures for D_1 and D_2.

From the structure of the sports competition intuitively one could argue that Team 3 has performed the best of these four teams. This is due to the fact that it beat the strong Team 2, while Team 2 only beat the weaker Team 1. (That team only drew with the weakest Team 4.) Note that the modified β-measure β' of representation D_2 is the only measure that reflects this intuitive sentiment.

This intuitive argument supplements the theoretical discussion in the previous section based on the fact that the modified β-measure satisfies node normalization rather than dominance normalization or degree normalization. Indeed, node normalization seems to be a very sensible requirement within sports competitions, in particular tournaments in which every team meets every other team for a match.

5.2.3 Iterated Power Measures

Next we consider the iterated application of a measure on a directed network to derive a related measure. The prime candidates for such iterated applications are the degree and β-measures. however, both have significant drawbacks due to the normalization properties these measures satisfy. In fact for iterated application it is optimal if the measure in question satisfies node normalization rather than another form of normalization or efficiency.

I illustrate this by iterating the β-measure itself.[7]

[7] The iterated application of the degree measure is impossible. The reason is that not a proper iterative process can be defined without significantly altering the degree measure itself. This is not the case for the β-measure for which one can device a very natural iterative process.

Let $D \in \mathcal{D}^N$ be some directed network. In the first round of application we simply arrive at $b^1 = \beta$. Suppose the iteration process has been applied $t - 1$ times and resulted into a measure b^{t-1}. Then the t-th iteration can be defined by an application of the β-measure characteristic formula to the network D in which all players $i \in N$ have a weight b_i^{t-1}. Hence, each iteration the β-measure is applied to the same directed network in which the participating players have a different weight.

Formally, for every directed network $D \in \mathcal{D}^N$ and every player $i \in N$ this leads to the following formulation:

$$b_i^1(D) = \beta_i(D) \tag{5.26}$$

$$b_i^t(D) = \sum_{j \in D(i)} \frac{b_i^{t-1}(D)}{\# D^{-1}(j)} \tag{5.27}$$

The main question is whether this system converges to a uniquely determined limit measure b^*. To illustrate the problems that emerge for this particular iterative process, I consider the directed network depicted in Fig. 5.5. In this network, the β-measure is given by $\beta(D) = (0, 1\frac{1}{2}, 1\frac{1}{2}, 0)$. The iterative process described above is based on the 4×4-transition matrix $B = [b_{ij}]$ given by

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $b_{ij} = \frac{1}{\# D^{-1}(j)}$ if $j \in D(i)$ and $b_{ij} = 0$ otherwise. Note here that $b^1(D) = \beta(D) = Be$ where $e = (1, 1, 1, 1)$ and $b^t(D) = B\, b^{t-1}(D)$.

From this I compute $b^2(D) = B\, b^1(D) = (0, 0, 1\frac{1}{2}, 0)$ and $b^3(D) = B\, b^2(D) = 0$. So, we conclude that $b^t(D) = 0$ for $t \geqq 3$. In particular this implies that the limit measure $b^*(D) = \lim_{t \to \infty} b^t(D) = 0$.

An alternative approach based on linear algebra is to determine the limit measure by solving the equation $b^* = Bb^*$ for the limit measure b^* directly. This requires the determination of the matrix $(I - B)$. In particular, we have to check whether the matrix $(I - B)$ has a non-trivial null space, since we have to solve $(I - B)b^* = 0$. First, we compute that

$$(I - B) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -\frac{1}{2} \\ 0 & -1 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From this computation it is easy to see that $(I - B)$ does not have a non-trivial null space. This implies in fact that zero is no eigenvalue of $(I - B)$ and, therefore, the zero measure is the unique limit measure for this iterative process.

5.2.3.1 The Iterated β-Measure

From the previous discussion it can only be concluded that we need to be careful in the selection of the measure that is to be iterated. It is expected that this measure has to satisfy node normalization, since otherwise the iterative process does not have to converge for every directed network.

This leads to the selection of the modified β-measure β' for the iteration process. The iterative process can now be formalized as follows:

$$\beta_i^1(D) = \beta_i'(D) \tag{5.28}$$

$$\beta_i^t(D) = \sum_{j \in D(i) \cup \{i\}} \frac{\beta_i^{t-1}(D)}{1 + \#D^{-1}(j)} \tag{5.29}$$

This iterative process for the directed network D is based on an $|N| \times |N|$ transition matrix given by $T_D = [t_{ij}]$ where for all $i, j \in N$: $t_{ij} = \frac{1}{1+\#D^{-1}(j)}$ if $j \in D(i) \cup \{i\}$ and $t_{ij} = 0$ otherwise. Now

$$\beta^t(D) = T_D\, \beta^{t-1}(D) = T_D^{t-1}\, \beta'(D) = T_D^t\, e, \tag{5.30}$$

where $e = (1, \ldots, 1) \in \mathbb{R}^{|N|}$ is the $|N|$-dimensional vector of ones. The next lemma summarizes the main properties of this iterative process that defines the iterated β-measure. For a proof of this assertion I refer to Section 5.3 in Borm et al. (2002).

Lemma 5.14 (Borm et al., 2002) *For every directed network $D \in \mathcal{D}^N$ the iterative process defined by (5.30) is a Markov chain which has a unique stationary distribution. Furthermore, $\beta^*(D) = \lim_{t\to\infty} \beta^t(D)$ exists and is uniquely determined.*

This lemma justifies the following definition.

Definition 5.15 The *iterated β-measure* is the unique measure $\beta^* \colon \mathcal{D}^N \to \mathbb{R}^N$ with for every directed network $D \in \mathcal{D}^N$:

$$\beta^*(D) = \lim_{t\to\infty} \beta^t(D) = \lim_{t\to\infty} T_D^t\, e.$$

Returning to the directed network D depicted in Fig. 5.5 I compute the transition matrix T for this particular network as

$$T = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 1 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

It is easy to compute that $Te = (\frac{1}{2}, 1\frac{1}{3}, 1\frac{5}{6}, \frac{1}{3}) = \beta'(D)$. From this it can also be determined that

$$\beta^1(D) = Te = \left(\tfrac{1}{2}, 1\tfrac{1}{3}, 1\tfrac{5}{6}, \tfrac{1}{3}\right)$$
$$\beta^2(D) = T^2e = \left(\tfrac{1}{4}, 1\tfrac{1}{36}, 2\tfrac{11}{18}, \tfrac{1}{9}\right)$$
$$\beta^3(D) = T^3e = \left(\tfrac{1}{8}, \tfrac{73}{108}, 3\tfrac{35}{216}, \tfrac{1}{27}\right)$$

From these computations it is clear that the iterative process puts ever more weight on player 3. Indeed, this player is the only undominated player in this directed network. This exceptional status results into the accumulation of power.

To correctly compute the iterated β-measure $\beta^*(D)$ for this directed network, I turn again to the computation of the solution to the equation $\beta^* = T\beta^*$ or $(I-T)\beta^* = 0$. For that purpose I note that

$$(I-T) = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

From this expression it is easy to compute that the null space of $(I-T)$ is spanned by a unique vector, namely $(0,0,1,0)$. Re-normalizing this spanning vector such that it satisfies node normalization, I arrive at $\beta^*(D) = (0,0,4,0)$. This is exactly what one should expect after the computations of $\beta^2(D)$ and $\beta^3(D)$.

5.2.3.2 The Positional Power Measure

Recently Herings, van der Laan, and Talman (2005) extended the discussion of power measures on directed networks to include a so-called "global" measure. Thus far the discussed measures are based only on the local aspects of the network, namely the (direct) successors and (direct) predecessors of players in the network. Of course, the β-measure of a player i is not completely local, but is unaffected by changes in the network beyond $D(i) \cup D^{-1}(i) \cup \{i\}$. As such it can still be denoted as non-global.

The aim of Herings et al. (2005) to introduce a truly global measure is founded on an iterated structure. First, I have to introduce some auxiliary concepts. Let $D \in \mathcal{D}^N$ be some directed network. The *adjacency matrix* for D is an $n \times n$-matrix A_D with entries $A_D(i,j) = 1$ if $j \in D(i)$ and $A_D(i,j) = 0$ otherwise.

Herings et al. (2005) first discuss the most plausible way to achieve a global measurement of power in directed networks. This *long-path method* of power measurement is based on the iterated application of the adjacency matrix. Indeed, define

$$x^0 = e \quad \text{and} \quad x^t = A_D\, x^{t-1}. \tag{5.31}$$

From this definition it follows that $x^1 = \delta(D)$ is the degree measure. The procedure defined above is said to converge if

$$\text{LP}(D) = \lim_{t\to\infty} \frac{x^t}{\sum_{i\in N} x_i^t} \tag{5.32}$$

exists. If this limit exists, then LP(D) is called the *long-path measure*. Of course, the long-path measure does neither have to exist nor have to be a non-trivial measure. For example, if D is a transitive tournament, then LP(D) is the zero vector, which is not very useful. The next lemma states exactly when the long-path measure is non-trivial:

Lemma 5.16 (Moon and Pullman, 1970) *Let $D \in \mathcal{D}^N$ be strongly connected in the sense that for every two players $i,j \in N$ there exists a sequence $i_1,\dots,i_K \in N$ such that $i_1 = i$, $i_K = j$ and $i_{k+1} \in D(i_k)$ for every $k = 1,\dots,K-1$. Then the long-path measure* LP(D) *exists and (up to normalization) is equal to the unique strictly positive eigenvector of the adjacency matrix A_D of network D.*

Usually the long-path measure is not useful if the directed network in question is not strongly connected.

Other global methods have been devised in the literature, but these measures have not resulted into very important innovations. I discuss the measure introduced by Herings et al. (2005) themselves. This so-called HLT-measure has interesting properties, but is nt completely analyzed yet; a complete axiomatization of the HLT-measure is yet missing from the literature on power measures.

The foundation of the HLT-measure is that the positional power of a player in a directed network is based on the number of its successors and the power of these successors. Thus, the power m_i of $i \in N$ is determined to be

$$m_i = \sum_{j\in D(i)} \left[\mu + \frac{1}{\nu} m_j \right]. \tag{5.33}$$

In matrix notation this is equivalent to

$$m = \mu\delta(D) + \tfrac{1}{\nu} A_D\, m \quad \text{or} \quad \left(I - \tfrac{1}{\nu} A_D \right) m = \mu\delta(D). \tag{5.34}$$

The (μ,ν)-*HLT-measure* is now the unique solution m of (5.34) if it exists.

Proposition 5.17 (Herings et al., 2005, Theorem 3.1) *For every directed network $D \in \mathcal{D}^N$ the (μ,ν)-HLT-measure exists and is nonnegative if $\mu \geqq 0$ and $\nu > n-1$.*

Moreover, for $\nu > n-1$ the matrix $\left(I - \frac{1}{\nu} A_D \right)$ has an inverse and all elements in this inverse are nonnegative.

Proof Let b_{ij} be the (i,j)-th element of the matrix $B = \left(I - \frac{1}{\nu} A_D \right)$. Since $b_{ii} = 1$ for all $i \in N$ and $b_{ij} \leqq 0$ for all $i \neq j$, the inverse of B exists and is nonnegative if and only if there exists some nonnegative vector $x \in \mathbb{R}^N$ such that $y = Bx \gg 0$.

Take $x = e$. Then $y_i = \sum_{j=1}^n b_{ij} = 1 - \sum_{j\in D(i)} \frac{1}{\nu} \geqq 1 - \frac{1}{\nu}(n-1)$. Hence, if $\nu \geqq n-1$, then $y_i > 0$ for all $i \in N$. Therefore, $y \gg 0$ and B has an inverse with nonnegative elements. Since $\mu \geqq 0$ as well as $\delta(D) \geqq 0$, it follows that (5.34) indeed has a unique nonnegative solution m^*, which is the desired (μ, ν)-HLT-measure of D. ■

The most plausible values for μ and ν are the minimal natural ones. Thus, $\mu = 1$ and $\nu = n$. The resulting $(1, n)$-HLT-measure is denoted as the positional power measure by Herings et al. (2005). The *positional power measure* can be formulated as

$$\rho(D) = \left(I - \frac{1}{n} A_D \right)^{-1} \delta(D) \tag{5.35}$$

I conclude the discussion of the positional power measure ρ with a listing of some of its main properties:

(i) For every $i \in N$: $\rho_i(D) > 0$ if and only if $D(i) \neq \varnothing$.
(ii) For every pair of players $i, j \in N$ it holds that $\rho_i(D) \geqq \rho_j(D)$ if $D(i) \subset D(j)$ with equality only if $D(i) = D(j)$.
(iii) Let D' be given by $D'(h) = D(h) \cup \{k\}$ and $D'(i) = D(i)$ for all $i \neq h$. Then the following hold:

- $\rho_i(D') \geqq \rho_i(D)$ for all $i \in N$;
- $\rho_h(D') - \rho_h(D) > \max_{i \neq h} \left(\rho_i(D') - \rho_i(D) \right)$, and
- For every $i \neq h$, $\rho_i(D') \neq \rho_i(D)$ if and only if there is no ordered path of any length in network D from player i to player h.

For a proof of these properties I refer to Section 5.4, in particular Lemma 4.4, in Herings et al. (2005). There one can also find the presentation of more properties of the positional power measure ρ and related concepts.

5.3 Hierarchical Allocation Rules on Network Games

Thus far I only considered the analysis of directed networks without the participating players having the abilities to generate values from cooperation through these network relationships. These are so-called domination networks. In this section I turn to the analysis of so-called "cooperative network situations" or simply information sharing networks. These are cooperative games within the context of a given directed network. The cooperation—or communication—structure on these coalitions is crucial for the specific value generated in these coalitions. In short, coalitional values are generated through the sharing of productive information within each coalition. The specific network communication structure within each coalition is here of particular importance. Thus, this implies that one has to formulate the generated values through a functions on the networks themselves. This is indeed considered in the current section.

The network is here given as a given set of communication relationships through which coalitions form in the sense of Myerson (1977). These communication relationships are supported by institutional structures that accommodate meaningful exchange of information between the players constituting these relationships. Mainly, players can only cooperate and exchange information through the network and as a consequence not every coalition can freely form and generate cooperation values. This implies that there results a certain structure of feasible coalitions within the given network. An allocation rule to be considered in this context is usually a Myerson-like value, already defined and explored in Chapter 3 of these lecture notes.

Within these cooperative network situations, players employ the available communication relationships in a well-defined fashion. In every communication link we distinguish an initiator and a respondent. The *initiator* explicitly makes a rational decision to initiate the communication of certain information. It might even be assumed that such initiation is costly and that the initiator has to invest effort to reach the other player that she wants to communicate with. The *respondent* replies to the initiated contact by an initiator. Usually there are no costs related to responding to communication efforts by other players.

Now a directed communication link is explicitly viewed as going from an initiator to a respondent. This implies a specific interpretation of the notion of "directed" communication. This perspective has natural consequences regarding the expectations that these two players have on the allocated benefits from communication. Indeed, an initiator naturally expects to have higher returns from such communicative activities than a respondent. This is fully taken into account in the discussion in this chapter on allocation rules for such directed communication situations.

My discussion of this cooperative game theoretic framework here is primarily based on Slikker, Gilles, Norde, and Tijs (2005), which in turn is inspired by and extends Myerson (1977). In particular I discuss plausible generalizations of the balanced payoff property, which is the corner stone of Myerson's approach. The main conclusion of the following analysis is that directed networks very much need to be considered as proto-hierarchies in which the depicted links represent authority relationships.

5.3.1 Cooperative Network Situations

I first introduce some auxiliary notation. Let $D \in \mathcal{D}^N$ be some directed network on player set N. Suppose $D' \in \mathcal{D}^N$ has the property that $D'(i) \subset D(i)$ for all $i \in N$, then we denote this by $D' \subset D$.

Suppose that $j \in D(i)$ for some $i, j \in N$. Then we denote by $(D - ij) \in \mathcal{D}^N$ the directed network given by $(D - ij)(h) = D(h)$ for all $h \neq i$ and $(D - ij)(i) = D \setminus \{j\}$. Thus, $(D - ij)$ denotes the directed network that is created by deleting the directed link between players i and j.

An undirected path or *u-path* between player i and j in directed network D is a finite sequence of players $P_c = (i_1, \ldots, i_m)$ such that $i_1 = i$, $i_m = j$ and for every

$k \in \{1, \dots, m-1\}$ either $i_{k+1} \in D(i_k)$ or $i_k \in D(i_{k+1})$. Clearly a u-path connects two players regardless of the direction of the individual links in this path. In this approach it is assumed that the direction of a link is not relevant for its communication properties; rather, the direction of a link indicates who initiated the link. Hence, if $i \in D(j)$, then j initiated the creation of the link with i, but the communication properties of this link are reciprocal.

We now say that two players $i, j \in N$ are *u-connected* in D if either $i = j$ or there exists a u-path between i and j. Players i and j are in the same *u-component* if and only if i and j are u-connected. This results into a partitioning of N into a finite number of u-components. This partitioning is indicated by $N/D = \{N_1, \dots, N_p\}$, where $p \geqq 1$ is the total number of u-components of D.

If $S \subset N$ is some coalition, then $D_S \in \mathcal{D}^S$ is the *S-restriction* of D on S given by $D_S(i) = D(i) \cap S$. Now S/D stands for the resulting partitioning of S into u-components of D_S, i.e., $S/D = S/D_S$.

Finally, let $\mathfrak{D} \subset \mathcal{D}^N$ be some collection of directed networks on N. The collection $\mathfrak{D}$ is *closed* if for all $D \in \mathfrak{D}$ and $D' \in \mathcal{D}^N$ with $D' \subset D$ it holds that $D' \in \mathfrak{D}$. With these preliminaries I can now introduce the main tool in our analysis, a (directed) cooperative network situation.[8]

Definition 5.18 Let $\mathfrak{D} \subset \mathcal{D}^N$ be some closed collection of directed networks on N.

(i) *A (network) benefit function* on $\mathfrak{D}$ is a function $r \colon \mathfrak{D} \to \mathbb{R}$ such that $r(D_\varnothing) = 0$ and r satisfies the *component additivity property* in the sense that for every directed network $D \in \mathfrak{D}$:

$$r(D) = \sum_{S \in N/D} r(D_S). \tag{5.36}$$

(ii) A triple (N, r, D) is a *cooperative network situation* on $\mathfrak{D}$ if r is a benefit function on $\mathfrak{D}$ and $D \in \mathfrak{D}$ is some status quo directed network. The collection of all cooperative network situations on $\mathfrak{D}$ is now denoted by $\mathbb{S}(\mathfrak{D})$.

(iii) An *allocation rule* on $\mathbb{S}(\mathfrak{D})$ is now a mapping $\gamma \colon \mathbb{S}(\mathfrak{D}) \to \mathbb{R}^N$ that assigns to every cooperative network situation $(N, r, D) \in \mathbb{S}(\mathfrak{D})$ a payoff vector $\gamma(N, r, D) \in \mathbb{R}^N$.

A cooperative network situation (N, r, D) clearly describes the generation of values from cooperation within the context of a directed communication network. First, the situation is defined within context of a certain closed collection of directed networks. This allows the deletion of any set of links in the directed network under consideration D. Second, the benefit function r describes how the cooperatively generated

[8] I point out the convention that by $D_\varnothing \in \mathcal{D}^N$ stands for the empty network given by $D_\varnothing(i) = \varnothing$ for all $i \in N$. Note here that the empty network is always a member of any closed collection of directed networks.

values change in response to changes in the network, in particular the deletion of certain links in the network.

From this perspective a cooperative network situation is truly a generalization of a standard cooperative game to the setting with directed networks. Later I will discuss this more explicitly.

5.3.2 Network Myerson Values

It is a simple step to generalize the Myerson value introduced for cooperative games with constraints on coalition formation to the setting of cooperative network situations. In this regard, a cooperative network situation is explicitly recognized as a cooperative game with constraints on coalition formation incorporated through the network structure that is imposed exogenously in the form of the status quo network.

This generalized Myerson value again satisfies generalizations of the two conditions discussed for the original Myerson value in Chapter 3, in particular Theorem 3.30. The properties introduced here generalize the component efficiency and balanced payoff properties discussed in Chapter 3.

Let γ be some allocation rule on $\mathbb{S}(\mathfrak{D})$ and let $w \in \mathbb{R}^N_{++}$ be some strictly positive weight vector. This rule can now satisfy the following properties:

u-Component Efficiency (CE) For every cooperative network situation $(N, r, D) \in \mathbb{S}(\mathfrak{D})$ and all u-components $S \in N/D$ it holds that

$$\sum_{i \in S} \gamma_i(N, r, D) = r(D_S). \tag{5.37}$$

w-Balanced Payoff Property (w-BPP) For any given cooperative network situation $(N, r, D) \in \mathbb{S}(\mathfrak{D})$ and all player pairs $i, j \in N$ with $j \in D(i)$ it holds that

$$\frac{\gamma_i(N, r, D) - \gamma_i(N, r, D - ij)}{w_i} = \frac{\gamma_j(N, r, D) - \gamma_j(N, r, D - ij)}{w_j} \tag{5.38}$$

Before I can formulate the main characterization theorem of the Myerson value for this setting, I need to introduce a few more concepts, which already should be familiar. First, for a strictly positive weight vector $w \in \mathbb{R}^N_{++}$ we define the *w-weighted Shapley value* on $\mathcal{G}^N$ by

$$\varphi_i^w(v) = \sum_{S \subset N:\ i \in S} \frac{w_i}{w(S)} \Delta_v(S), \tag{5.39}$$

where $w(S) = \sum_{j \in S} w_j > 0$ is the total weight assigned to coalition S. The weighted Shapley value has been investigated thoroughly by Kalai and Samet (1988); they provide a complete characterization of this family of values.

Second, every cooperative network situation $(N, r, D) \in \mathbb{S}(\mathfrak{D})$ can be converted into a corresponding cooperative game $v^{r,D} \in \mathcal{G}^N$ given by

$$v^{r,D}(S) = r(D_S). \tag{5.40}$$

The proof of the next theorem should by now be rather familiar; a proof is therefore relegated to the appendix of this chapter.

Theorem 5.19 *Let $\mathfrak{D} \subset \mathcal{D}^N$ be a closed collection of directed networks and $w \in \mathbb{R}^N_{++}$ be some strictly positive weight vector. Then there exists exactly one allocation rule on $\mathbb{S}(\mathfrak{D})$ that satisfies u-component efficiency and the w-balanced payoff property and it is given by the w-**Network Myerson Value** μ^w defined by*

$$\mu^w(N, r, D) = \varphi^w\left(v^{r,D}\right). \tag{5.41}$$

The Network Myerson value concept is a direct and immediate extension of the (weighted) Shapley value concept to the setting of cooperative network situations. The weights used in the w-balanced payoff property, however, seem to be unnatural and "forced". It would be more appropriate if these weights had an endogenous origin resulting from the position of a player in the network. This is the next stage in the development of this approach.

Example 5.20 Consider the directed network D given in Fig. 5.8 on the player set $N = \{1, 2, 3, 4\}$. The network D is given by $D(1) = \{2\}$, $D(2) = \{3\}$, $D(3) = \varnothing$, and $D(4) = \{3\}$. Furthermore, consider the closed collection of networks given by $\mathfrak{D} = \{D' \mid D'(i) \subset D(i) \text{ for all } i \in N\}$. I also consider the network benefit function r on $\mathfrak{D}$ given by $r(D)$ being the number of links in the network D. The network situation (N, r, D) can now be represented by the following table:

S	$r(D_S)$	$\Delta_{v^{r,D}}$
$\{i\}$	0	0
$\{1,2\}$	1	1
$\{1,3\}$	0	0
$\{1,4\}$	0	0
$\{2,3\}$	1	1
$\{2,4\}$	0	0
$\{3,4\}$	1	1
$\{1,2,3\}$	2	0
$\{1,2,4\}$	1	0
$\{1,3,4\}$	1	0
$\{2,3,4\}$	2	0
N	3	0

I first consider the straightforward application of the weight vector $w = e = (1, 1, 1, 1)$, leading to the regular Network Myerson value. From the third column of the table above, I deduce easily that $\mu^e(N, r, D) = \left(\frac{1}{2}, 1, 1, \frac{1}{2}\right)$.

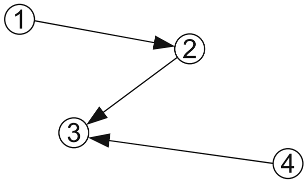

Fig. 5.8 Network discussed in Example 5.20

Next, consider the β-measure on the network D given by $\beta(D) = \left(1, \frac{1}{2}, 0, \frac{1}{2}\right)$ and use that measure as a weight vector on the cooperative network situation (N, r, D). For the weight vector $\beta(D)$ the relevant coalitional weights[9] are given by $\beta(12) = 1\frac{1}{2}$ and $\beta(23) = \beta(34) = \frac{1}{2}$. This results into a corresponding β-Network Myerson value computed as $\mu^{\beta}(N, r, D) = \left(\frac{2}{3}, 1\frac{1}{3}, 0, 1\right)$. ■

5.3.3 The Hierarchical Payoff Property

As discussed above, a directed communication link in a directed network D from player i to player j, i.e., $j \in D(i)$, can be interpreted that player i takes the initiative to form the communication relationship with player j, while the latter has a more passive role. Hence, player i can be interpreted as the *initiator* of the link with j, while player j can be interpreted as the *respondent* in the formation of the communication relationship between i and j. This is exactly the interpretation that I will follow in this section.

It seems natural that the initiator of a communication link expects a higher share of the generated benefits—as a "reward" for forming that link—than the respondent. So, although the communication relationship can be viewed as undirected in its value generating effects, it has been formed at the initiative of only one of the two players involved and that initiative might require a reward.

This expectation of the initiator can be formulated as a formal property of the allocation rule under consideration:

Definition 5.21 Let $\mathfrak{D} \subset \mathcal{D}^N$ be some closed collection of directed networks. An allocation rule γ on the corresponding collection of cooperative network situations $\mathbb{S}(\mathfrak{D})$ satisfies the *Hierarchical Payoff Property (HPP)* if for every $(N, r, D) \in \mathbb{S}(\mathfrak{D})$ and $i, j \in N$ with $j \in D(i)$ it holds that

(i) If $\gamma_i(N, r, D) > \gamma_i(N, r, D - ij)$, then

$$\gamma_i(N, r, D) - \gamma_i(N, r, D - ij) > \gamma_j(N, r, D) - \gamma_j(N, r, D - ij) \geqq 0.$$

[9] These are the weights for the coalitions with a non-zero Harsanyi dividend.

(ii) If $\gamma_i(N, r, D) = \gamma_i(N, r, D - ij)$, then

$$\gamma_j(N, r, D) = \gamma_j(N, r, D - ij).$$

(iii) If $\gamma_i(N, r, D) < \gamma_i(N, r, D - ij)$, then

$$\gamma_i(N, r, D) - \gamma_i(N, r, D - ij) < \gamma_j(N, r, D) - \gamma_j(N, r, D - ij) \leqq 0.$$

The following example shows that in general there does not exist an allocation rule that satisfies u-component efficiency as well as the hierarchical payoff property on an arbitrary closed collection of directed networks.

Example 5.22 Consider $N = \{1, 2, 3\}$ and the cyclic network D^* on N given by $D^*(1) = \{2\}$, $D^*(2) = \{3\}$ and $D^*(3) = \{1\}$. This network is depicted in Fig. 5.9. Now consider the closed collection of directed networks $\mathfrak{D} = \{D \mid D \subset D^*\}$. Furthermore, I consider the benefit function $r\colon \mathfrak{D} \to \mathbb{R}$ given by $r(D^*) = 1$ and $r(D) = 0$ for all other directed networks $D \neq D^*$.

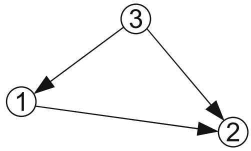

Fig. 5.9 The 3-player cyclic network D^*

Suppose that γ is an allocation rule on $\mathbb{S}(\mathfrak{D})$ that satisfies CE. It can now be shown that γ cannot satisfy HPP.

Assume to the contrary that γ actually satisfies HPP. First, it is clear that from CE, $\gamma_i(N, r, D_\varnothing) = 0$ for all $i \in N$. Next consider D consisting of a single link, i.e., $D(i) = \{j\}$ and $D(h) = \varnothing$ for $h \neq i$. Then by CE, $\gamma_h(D) = 0$ for $h \notin \{i, j\}$. Furthermore, by HPP if $\gamma_i(D) > 0$, then $\gamma_i(D) > \gamma_j(D) \geqq 0$ and, if $\gamma_i(D) < 0$, then $\gamma_i(D) < \gamma_j(D) \leqq 0$. Thus, by CE, $\gamma_i(D) + \gamma_j(D) = r(D) = 0$ and, therefore, $\gamma_i(D) = \gamma_j(D) = 0$.

With similar arguments one can show that $\gamma_i(D) = 0$ for all directed networks $D \subsetneq D^*$ and all players $i \in N$.

Finally, consider D^* itself. From CE it follows that $\gamma_i(D^*) > 0$ for some player $i \in N$. Without loss of generality assume that $\gamma_1(D^*) > 0$. By HPP, it then follows that

$$\begin{aligned}
\gamma_1(D^*) - \gamma_1(D^* - 12) &> \gamma_2(D^*) - \gamma_2(D^* - 12) \geqq 0 \\
\gamma_2(D^*) - \gamma_2(D^* - 23) &\geqq \gamma_3(D^*) - \gamma_3(D^* - 23) \geqq 0 \\
\gamma_3(D^*) - \gamma_3(D^* - 31) &\geqq \gamma_1(D^*) - \gamma_1(D^* - 31) \geqq 0
\end{aligned}$$

Using $\gamma_i(D) = 0$ for all directed networks $D \subsetneq D^*$ and all players $i \in N$, we get

$$\gamma_1(D^*) > \gamma_2(D^*) \geqq \gamma_3(D^*) \geqq \gamma_1(D^*).$$

This is a contradiction. Hence, one can only conclude that there is no allocation rule on $\mathbb{S}(\mathfrak{D})$ that satisfies CE as well as HPP. From the construction of this example is clear that the cyclic nature of the network D^* is essential here. ■

The example shows that cycles are hard to deal with if we impose the hierarchical payoff property or variations thereof. Next I discuss acyclicity requirements that exclude these cycles from being present in the networks under consideration.

Consider a u-path $P = \{i_1, \dots, i_m\}$ in some directed network $D \in \mathcal{D}^N$. The *regularity index* of the path P is now defined by

$$t(P) = \#\{k \mid i_{k+1} \in D(i_k)\} - \#\{k \mid i_k \in D(i_{k+1})\}. \tag{5.42}$$

So, the regularity index of a path indicates the difference in the number of links in one direction and the other directed. In particular, if $t(P) = m - 1$ or $t(P) = -m + 1$, then the links in path P are directed in the same direction, i.e., the path P is "directed".

Definition 5.23 Let $D \in \mathcal{D}^N$ be some directed network.

(i) The network D is *acyclic* if for every cyclic path $C = \{i_1, \dots, i_m\}$ with $i_1 = i_m$ it holds that $-m + 2 \leqq t(C) \leqq m - 2$.
(ii) The network D is *weakly hierarchical* if there exists an ordered partitioning $\mathcal{H} = \{H_1, \dots, H_p\}$ of the player set N such that for all pairs $i, j \in N$ with $j \in D(i)$ there exist $1 \leqq k_j < k_i \leqq p$ with $i \in H_{k_i}$ and $j \in H_{k_j}$.
The class of all weakly hierarchical networks is denoted by $\mathfrak{H}_w^N \subset \mathcal{D}^N$.

Acyclicity and weak hierarchies are already discussed in Chapter 2 in the context of constraints on coalition formation. Acyclicity excludes directed cycles in the network; as should be clear from the definition, cycles that are directed are exactly excluded, i.e., for a cycle C it cannot hold that $t(C) = m - 1$ or $t(C) = -m + 1$.

The next theorem formally links the definitions of acyclicity of directed networks and of weak hierarchies with the definition of strict permission structures discussed in Definition 2.30. A proof is relegated to the appendix of the current chapter.

Theorem 5.24 *Let $D \in \mathcal{D}^N$ be some directed network. Then the following statements are equivalent:*

(i) *D is an acyclic directed network.*
(ii) *D is a strict permission structure.*
(iii) *D is weakly hierarchical.*

That the collection of weakly hierarchical networks indeed has the desired properties to allow the hierarchical payoff property is subject to the following theorem that has wide-ranging consequences. For a proof I refer to the appendix of this chapter.

Theorem 5.25 *Let N be a finite set of players.*

(a) *Let $\mathfrak{D} \subset \mathfrak{H}_w^N$ be some closed collection of weakly hierarchical networks on N. Then there exists at least one allocation rule on $\mathbb{S}(\mathfrak{D})$ that satisfies both component efficiency as well as the hierarchical payoff property.*
(b) *Let $D^* \notin \mathfrak{H}_w^N$ be a non-weakly hierarchical network and let $\mathfrak{D} \subset \mathcal{D}^N$ be a closed collection of networks such that $D^* \in \mathfrak{D}$. Then there is no allocation rule on $\mathbb{S}(\mathfrak{D})$ that satisfies both component efficiency and the hierarchical payoff property.*

This theorem has some profound consequences. It essentially states that the hierarchical payoff property can only be applied to weakly hierarchical networks. Thus, only networks without any directed cycles can support allocation rules that satisfies the HPP.

This in turn implies that, if rewards to the initiation of such beneficial relationships are expected, the hierarchical payoff property is invoked. The main insight now states that such a reward system can only be implemented within hierarchically structured networks. Hence, this theorem gives us an understanding why in business there are such hierarchies applied to the organization of production processes. Indeed, if an owner of a production technology expects a return on his asset that exceeds the return to an employee using that asset, then it can only occur within a hierarchical organization. In such an organization subordinates have proper access to the productive asset, without jeopardizing the higher return on this asset for the owner.

In this light, the main result stated on the hierarchical payoff property informs us why the hierarchical organization form is so prevalent in the capitalist society. It can achieve allocations that cannot be accomplished in the context of a market allocation mechanism which by nature is more egalitarian. For an elaborate illustration of this main insight I refer to Example 6.21 in the next chapter on hierarchical structures. There the comparison of a market mechanism with a hierarchical production organization is developed further.

5.3.4 The α-Hierarchical Value

The hierarchical payoff property imposes that the initiator in a relationship receives a higher return from this relationship than the respondent. It does *not* state how much higher this return is. Next, I strengthen the hierarchical payoff property by requiring that the initiator receives a certain factor of the respondents benefit from the added relationship. This factor is assumed to be determined exogenously and given by $\alpha > 0$.

Definition 5.26 Let $\mathfrak{D} \subset \mathcal{D}^N$ be some closed collection of directed networks and let $\alpha > 0$. An allocation rule γ on the corresponding collection of cooperative network situations $\mathbb{S}(\mathfrak{D})$ satisfies the *α-Hierarchical Payoff Property* (*α-HPP*) if for every $(N,r,D) \in \mathbb{S}(\mathfrak{D})$ and $i,j \in N$ with $j \in D(i)$ it holds that

$$\gamma_i(N,r,D) - \gamma_i(N,r,D-ij) = \alpha \left[\gamma_j(N,r,D) - \gamma_j(N,r,D-ij) \right] \tag{5.43}$$

The exogenously factor $\alpha > 0$ is crucial in interpreting the meaning of this property. If $\alpha = 1$ we arrive exactly at the balanced payoff property considered before in the context of the Myerson value. This results into a regular or "fair" Myerson payoff structure.

On the other hand, if $\alpha > 1$ we arrive at the natural requirement that the initiator receives a higher benefit—or loss for that matter—than the respondent from the addition of the link ij to the network. This seems to be the most natural expression of a quantification of the hierarchical payoff property.

Finally, for $\alpha < 1$, we arrive at exactly the opposite property. In that case the initiator receives less than the respondent from the creation of an additional relationship in the network. This seems rather unnatural, and therefore it is normal to assume that $\alpha \geqq 1$.

The next analysis shows that the α-Hierarchical Payoff Property can only be used in the context of certain specific networks. These networks have a property that is stronger than the weakly hierarchical structure imposed in the previous discussion.

Definition 5.27 Let $D \in \mathcal{D}^N$ be some directed network.

(i) The network D is *cycle regular* if for every cyclic path $C = \{i_1, \dots, i_m\}$ with $i_1, \dots, i_{m-1}$ distinct and $i_1 = i_m$ it holds that $t(C) = 0$.
(ii) The network D is *hierarchical* if there exists an ordered partitioning $\mathcal{H} = \{H_1, \dots, H_p\}$ of the player set N such that for all pairs $i,j \in N$ with $j \in D(i)$ there exists $1 \leqq k \leqq p-1$ with $i \in H_{k+1}$ and $j \in H_k$.
The class of all hierarchical networks is denoted by $\mathfrak{H}^N \subset \mathfrak{H}^N_w \subset \mathcal{D}^N$.

Cycle regularity imposes that all cycles in the network have the same regularity index, namely zero. Hence, in every cycle the number of links pointing one directed is exactly equal to the number of links pointing the other direction. This also implies that all cycles have even length.

The difference between weakly hierarchical and (regularly) hierarchical networks is the following. Weakly hierarchical networks have a tiered or layered structure in which all links go from a higher layer to a lower layer. In a hierarchical network there is a structure of layers such that links only exist between subsequent layers or tiers. So, links are only directed from a player in a certain tier to players in the tier that is just below the tier of that player.

As one can expect, these two properties are in fact equivalent.

Proposition 5.28 *Let $D \in \mathcal{D}^N$ be some directed network. Then D is cycle regular if and only if D is hierarchical.*

Proof First, suppose that D is cycle regular. We assign an index number to each player. Let $C \in N/D$ be a u-component of D and let $i \in C$ be fixed. Set $P_i^C = 0$.

For all $j \in C\backslash\{i\}$ consider an arbitrary, but fixed path from i to j, $P_{ij} = (i_1, \ldots, i_m)$ with $i_1 = i$ and $i_m = j$. Now define $p_j^C = -t(P_{ij})$. By cycle regularity it is clear that the choice of p_j^C is independent of the path P_{ij} selected. Furthermore, note that by construction for all $j, h \in C$ if $h \in D(j)$, then $p_j^C = p_j^C + 1$. Indeed take a path P_{ij} from i to j, then $P_{ih} = (P_{ij}, h)$ is a path from i to h. Obviously, $t(P_{ih}) = t(P_{ij}) + 1$ implying that $p_j^C = -t(P_{ij}) = -t(P_{ih}) + 1 = p_h^C + 1$.

In this fashion p_j^C is constructed for all u-components $C \in N/D$ and all players $j \in C$. Subsequently, define

$$H_k^C = \{j \in C \mid p_j^C = \min_{h \in C} p_h^C - 1 + k\} \tag{5.44}$$

for all $k \in \{1, \ldots, m^*\}$ where

$$m^* = \max_{C \in N/D} \left[\max_{h \in C} p_h^C - \min_{h \in C} p_h^C + 1 \right]. \tag{5.45}$$

Now, for all $j, h \in N$ it holds that, if $h \in D(j)$ and $j \in H_{k+1}^C$ for some u-component $C \in N/D$ and $k \in \{1, \ldots, m^* - 1\}$, then $h \in H_k^C$, since $p_h^C = p_j^C + 1$.

Therefore, $(H_1, \ldots, H_{m^*})$ with $H_k = \cup_{C \in N/D} H_k^C$ for $k \in \{1, \ldots, m^* - 1\}$ is an ordered partitioning of N with the desired property.

Second, suppose that D is hierarchical. Then there exists an ordered partitioning $\mathcal{H} = \{H_1, \ldots, H_p\}$ of N such that for all $i, j \in N$ with $j \in D(i)$ there exists $1 \leqq k \leqq p - 1$ with $i \in H_{k+1}$ and $j \in H_k$.

Now consider a cycle $C = (i_1, \ldots, i_{m+1})$ with $i_1 = i_{m+1}$. For every $r \in \{1, \ldots, m\}$ with $i_r \in H_k$ it holds that, if $i_r \in D(i_{r+1})$, then $i_{r+1} \in H_{k-1}$, and if $i_{r+1} \in D(i_r)$, then $i_{r+1} \in H_{k+1}$. Since $i_1 = i_{m+1}$, these belong to the same hierarchical tier, implying that

$$\#\{k \mid i_k \in D(i_{k+1})\} = \#\{k \mid i_{k+1} \in D(i_k)\}$$

This in turn implies that the cycle C indeed satisfies the cycle regularity property that $t(C) = 0$. Since C was arbitrarily chosen, this completes the proof that D is cycle regular. ■

Our next main result states unequivocally that allocation rules satisfying α-HPP only exist on collections of hierarchical networks.

Theorem 5.29 *Let $\alpha > 1$ and let $\mathfrak{D} \subset \mathcal{D}^N$ be some closed collection of directed networks.*

(a) *If $\mathfrak{D} \subset \mathfrak{H}^N$ is a closed collection of hierarchical networks, then there exists a unique allocation rule on $\mathbb{S}(\mathfrak{D})$ that satisfies component efficiency and the α-hierarchical payoff property.*

(b) *Let* $D^* \notin \mathfrak{H}^N$ *be a non-hierarchical network. If* $\mathfrak{D}$ *is such that* $D^* \in \mathfrak{D}$*, then there is no allocation rule on* $\mathbb{S}(\mathfrak{D})$ *that satisfies component efficiency as well as the* α*-hierarchical payoff property.*

This result effectively argues for the existence of a unique allocation rule on the collection of hierarchical networks that satisfies component efficiency as well as the α-hierarchical payoff property. This unique value can be indicated as the α-*hierarchical value* and is directly related to the weighted Network Myerson values discussed before. Unfortunately, although the α-hierarchical value is unique, it is rather cumbersome to construct and a closed definition cannot be formulated readily. Only a construction can be given for this value. For details of this construction I refer to the proof of Theorem 5.29 in the appendix to this chapter.[10]

I emphasize that this α-hierarchical value only exists for networks in which players can be placed in a hierarchical tier system. Thus, the implementation of the α-hierarchical payoff property requires a stronger property than the weak hierarchical network property that was imposed for the implementation of the (regular) hierarchical payoff property.

I conclude this chapter with a computational illustration of the α-hierarchical value for the directed network represented in Fig. 5.8 and the cooperative network situation discussed in Example 5.20.

Example 5.30 Again consider the directed network D on $N = \{1, 2, 3, 4\}$, the network benefit function r, and the closed collection of directed networks $\mathfrak{D} = \{D' \mid D'(i) \subset D(i) \text{ for all } i \in N\}$ given in Example 5.20. Note that the network D is indeed hierarchical.

Let $\alpha > 1$ be given. In this particular simple example we can now rather easily determine the corresponding α-hierarchical value for (N, r, D). The α-hierarchical value on D now corresponds to the w_α-Network Myerson value where the weight vector w_α is selected as $w_\alpha = (\alpha^2, \alpha, 1, \alpha)$. This weight vector is completely determined by the hierarchical position of a player in D; more precisely by the tier in D of which the player is a member.

Indicate the α-hierarchical value by h^α, then we compute that

$$h^\alpha(N, r, D) = \mu^{w_\alpha}(N, r, D) = \left(\frac{\alpha}{\alpha+1}, 1, \frac{2}{\alpha+1}, \frac{\alpha}{\alpha+1} \right).$$

Remark that for $\alpha = 1$ this exactly corresponds to the regular Network Myerson value computed in Example 5.20. This confirms the computations, since this is exactly what should be expected. ■

[10] I emphasize that the construction developed in the proof of Theorem 5.29 imposes that the α-hierarchical value is different for different collections of directed network. This is certainly not the case for the Network Myerson values discussed previously.

5.4 Appendix: Proofs of the Main Theorems

Proof of Theorem 5.4

It is easy to verify that the degree measure δ satisfies the four stated properties in Axiom 5.3.

To show the reverse, let m be a measure on $\mathcal{D}^N$ that satisfies the four properties stated in Axiom 5.3.

Let $D \in \mathcal{D}^N$ be some directed network. Now for every $j \in N$ define

$$D_j = \{(i,j) \in N \times N \mid i \in D^{-1}(j)\} \subset D$$

Now from the dummy position property and symmetry it can be concluded that for $j \in N$ there exists some constant $c_j \in \mathbb{R}$ such that

$$m_i(D_j) = \begin{cases} c_j & \text{if } i \in D^{-1}(j) \\ 0 & \text{otherwise.} \end{cases}$$

Since $\#D_j = \#D^{-1}(j)$ it follows from degree normalization that

$$c_j = \begin{cases} 1 & \text{if } D^{-1}(j) \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

The collection $\{D_j \mid j \in N\}$ forms an independent partition of D. From additivity over independent partitions it thus follows that for every $i \in N$:

$$m_i(D) = \sum_{j \in N} m_i(D_j) = \sum_{j \in D(i)} 1 = \delta_i(D).$$

This shows the assertion.

Proof of Theorem 5.6

It is easy to verify that the β-measure satisfies the four listed properties.

Let m be a measure that satisfies the four properties. We use the same proof technique as employed in the proof of Theorem 5.4.

Let $D \in \mathcal{D}^N$ be some directed network. Now for every $j \in N$ define

$$D_j = \{(i,j) \in N \times N \mid i \in D^{-1}(j)\} \subset D$$

Now from the dummy position property and symmetry it can be concluded that for $j \in N$ there exists some constant $c_j \in \mathbb{R}$ such that

$$m_i(D_j) = \begin{cases} c_j \text{ if } i \in D^{-1}(j) \\ 0 \text{ otherwise.} \end{cases}$$

Since $\#\{h \in N \mid D_j^{-1}(h) \neq \varnothing\} = 1$ if $D^{-1}(j) \neq \varnothing$ and $\#\{h \in N \mid D_j^{-1}(h) \neq \varnothing\} = 0$ if $D^{-1}(j) = \varnothing$, it follows from dominance normalization that

$$c_j = \begin{cases} \frac{1}{\#D^{-1}(j)} & \text{if } D^{-1}(j) \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{D_j \mid j \in N\}$ is an independent partition of D, it follows from additivity over independent partitions that

$$m_i(D) = \sum_{j \in N} m_i(D_j) = \sum_{j \in D(i)} \frac{1}{\#D^{-1}(j)} = \beta_i(D),$$

for all $i \in N$. This shows the assertion.

Proof of Theorem 5.10

Let $D \in \mathcal{D}^N$ be some directed network. Again as in the proofs above for every $j \in N$ let

$$D_j = \{(i,j) \in N \times N \mid i \in D^{-1}(j)\} \subset D$$

Then it is clear from the definition of p_D that

$$p_D(S) = \sum_{j \in N} p_{D_j}(S), \quad S \subset N,$$

where for $j \in N$

$$p_{D_j}(S) = \begin{cases} 1 & \text{if } S \cap D^{-1}(j) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

Let $i \in N$. Then

$$p_{D_j}(S \cup \{i\}) - p_{D_j}(S) = \begin{cases} 1 & \text{if } j \in D(i) \text{ and } S \cap D^{-1}(j) = \varnothing \\ 0 & \text{otherwise} \end{cases}$$

If $D^{-1}(j) \neq \varnothing$, then there are $\frac{|N|!}{\#D^{-1}(j)}$ permutations of the players in N such that $i \in D^{-1}(j)$ precedes all other predecessors of j in the network D_j. But then by the marginal value formulation of the Shapley value of i in the game p_{D_j}

$$\varphi_i(p_{D_j}) = \begin{cases} \frac{1}{\#D^{-1}(j)} & \text{if } j \in D(i) \\ 0 & \text{otherwise} \end{cases}$$

Hence, it now follows immediately that

$$\varphi_i(p_D) = \sum_{j \in N} \varphi_i(p_{D_j}) = \sum_{j \in D(i)} \frac{1}{\#D^{-1}(j)} = \beta_i(D).$$

This shows the first part of the assertion. To show the second part, we note that since $s_D = \sum_{j \in \bar{D}(N)} u_{D^{-1}(j)}$ it holds that

$$\varphi_i(s_D) = \sum_{j \in D(i)} \varphi_i\left(u_{D^{-1}(j)}\right) = \sum_{j \in D(i)} \frac{1}{\#D^{-1}(j)} = \beta_i(D).$$

Proof of Theorem 5.19

Throughout this proof let $w \in \mathbb{R}^N_{++}$ be some strictly positive weight vector and let $\mathfrak{D} \subset \mathcal{D}^N$ be some closed collection of directed networks.

In this proof we use the main result of Kalai and Samet (1988). They show that the w-Shapley value satisfies the dummy player property, additivity and the so-called partnership consistency property. A coalition $S \subset N$ is a *partnership* in the cooperative game $v \in \mathcal{G}^N$ if for all $T \subset S$ and $R \subset N \setminus S$: $v(R \cup T) = v(R)$. Hence, a partnership is an inseparable group of players who can only generate value together. Now the allocation rule ϕ is *partnership consistent* if for every partnership S in v it holds for every $i \in S$ that

$$\phi_i(v) = \phi_i(\phi_S(v)\, u_S) \quad \text{where} \phi_S(v) = \sum_{j \in S} \phi_j(v). \tag{5.46}$$

Using the Kalai–Samet characterization of φ^w we can now show the following:

Claim 5.31 *The w-Network Myerson value μ^w satisfies CE and w-BPP.*

Proof First we show that μ^w satisfies CE. Let $(N, r, D) \in \mathbb{S}(\mathfrak{D})$ and let $C \in N/D$ be a u-component of D. Define two cooperative games v^C and v^{-C} given by

$$v^C(T) = r(D_{T \cap C})$$
$$v^{-C}(T) = r(D_{T \setminus C})$$

Since C is a u-component of D and r is component additive, it holds that $v^{r,D} = v^C + v^{-C}$.

Since all $i \in C$ are dummies in v^{-C} by Kalai–Samet $\varphi_i^w(v^{-C}) = 0$. Similarly for $j \in N \setminus C$ we get that $\varphi_j^w(v^C) = 0$. Using additivity of the w-Shapley value as well as its efficiency we thus arrive at

$$\begin{aligned}\sum_{i \in C} \varphi_i^w(v^{r,D}) &= \sum_{i \in C} \varphi_i^w(v^C) + \sum_{i \in C} \varphi_i^w(v^{-C}) \\ &= \sum_{i \in C} \varphi_i^w(v^C) = \sum_{i \in N} \varphi_i^w(v^C) \\ &= v^C(N) = r(D_C).\end{aligned}$$

This shows indeed that μ^w satisfies CE.

Second, we show that μ^w satisfies w-BPP. Assume that $j \in D(i)$. Let $D' = (D - ij)$ and $v' = v^{r,D} - v^{r,D'}$. For all $T \subset N$ with $\{i,j\} \nsubseteq T$ it then follows

$$v'(T) = r(D_T) - r(D'_T) = 0 \quad \text{since } D_T = D'_T.$$

This means that $\{i,j\}$ is a partnership in v'. From partnership consistency of φ^w it then follows that

$$\varphi_i^w(v') = \varphi_i^w\left(\left(\varphi_i^w(v') + \varphi_j^w(v')\right) u_{\{i,j\}}\right) = \tfrac{w_i}{w_i + w_j}\left[\varphi_i^w(v') + \varphi_j^w(v')\right]$$

Similarly, $\varphi_j^w(v') = \frac{w_j}{w_i + w_j}\left[\varphi_i^w(v') + \varphi_j^w(v')\right]$. Thus, $\frac{\varphi_i^w(v')}{w_i} = \frac{\varphi_j^w(v')}{w_j}$. From this we then conclude that

$$\frac{\mu_i^w(N,r,D) - \mu_i^w(N,r,D')}{w_i} = \frac{\varphi_i^w(v')}{w_i} = \frac{\varphi_j^w(v')}{w_j} = \frac{\mu_j^w(N,r,D) - \mu_j^w(N,r,D')}{w_j}$$

where the first and third equalities follow from the definition of v' and additivity of φ^w. This indeed shows that μ^w satisfies w-BPP. ■

The next step in the proof of Theorem 5.19 is to show that there exists at most one allocation rule that satisfies CE and w-BPP.

Claim 5.32 *There is at most one allocation rule on $\mathbb{S}(\mathfrak{D})$ that satisfies CE as well as w-BPP.*

Proof Suppose that there are two allocation rules γ^1 and γ^2 on $\mathbb{S}(\mathfrak{D})$ that satisfy CE and w-BPP. We show that $\gamma^1 = \gamma^2$ by induction on the number of links.

First, let $(N, r, D) \in \mathbb{S}(\mathfrak{D})$ with $\#D = 0$, i.e., $D = D_\varnothing$. From CE it then is immediate that $\gamma^1(N, r, D) = \gamma^2(N, r, D) = 0$.

Second, let $p \geqq 1$ and suppose that $\gamma^1(N, r, D') = \gamma^2(N, r, D')$ for all $(N, r, D') \in \mathbb{S}(\mathfrak{D})$ with $\#D' \leqq p - 1$. Now let $(N, r, D) \in \mathbb{S}(\mathfrak{D})$ with $\#D = p$.

Suppose that $j \in D(i)$. From w-BPP of γ^1 we then get that

$$\tfrac{1}{w_i}\left(\gamma_i^1(D)-\gamma_i^1(D-ij)\right)=\tfrac{1}{w_j}\left(\gamma_j^1(D)-\gamma_j^1(D-ij)\right).$$

Using the induction hypothesis and w-BPP of γ^2 we then arrive at

$$\begin{aligned}w_j\,\gamma_i^1(D)-w_i\,\gamma_j^1(D)&=w_j\,\gamma_i^1(D-ij)-w_i\,\gamma_j^1(D-ij)\\&=w_j\,\gamma_i^2(D-ij)-w_i\,\gamma_j^2(D-ij)\\&=w_j\,\gamma_i^2(D)-w_i\,\gamma_j^2(D)\end{aligned}$$

Thus,

$$\frac{\gamma_i^1(D)-\gamma_i^2(D)}{w_i}=\frac{\gamma_j^1(D)-\gamma_j^2(D)}{w_j}.$$

This expression is also valid for all pairs $i,j\in N$ in the same u-component of D.

Let $i\in N$ and let $C\in N/D$ with $i\in C$. For all $j\in C$ we then have that

$$\frac{1}{w_i}\left[\gamma_i^1(D)-\gamma_i^2(D)\right]=\frac{1}{w_j}\left[\gamma_j^1(D)-\gamma_j^2(D)\right]=\Delta_C.$$

So, for all $j\in C$ we then have that

$$\gamma_j^1(D)-\gamma_j^2(D)=w_j\,\Delta_C.$$

Now CE of γ^1 and γ^2 implies that

$$\sum_{j\in C}\gamma_j^1(D)=\sum_{j\in C}\gamma_j^2(D)=r(D_C).$$

Thus,

$$w(C)\cdot\Delta_C=\sum_{j\in C}w_j\,\Delta_C=\sum_{j\in C}\left(\gamma_j^1(D)-\gamma_j^2(D)\right)=0.$$

Since $w(C)>0$, $\Delta_C=0$ and, so, $\gamma^1(D)=\gamma^2(D)$ on C. Since i and C were chosen arbitrarily, we conclude that $\gamma^1=\gamma^2$ in general. ∎

Proof of Theorem 5.24

The equivalence of (i) and (ii) is rather trivial to show. I refer to the definition of a strict permission structure given in Definition 2.30 for the details.

I will limit myself here to show that statements (i) and (iii) are equivalent. Note that it is obvious that every weakly hierarchical network is acyclic. I only have to show that every acyclic network is also weakly hierarchical.

Let $D \in \mathcal{D}^N$ be acyclic. Define for any $N' \subset N$ and $D' \subset D$

$$N^0(N', D') = \{i \in N' \mid \text{There is no } j \in N' \text{ with } i \in D'(j)\} \tag{5.47}$$

as the collection of players that have no predecessors in (N', D'). Since D is acyclic it has to hold that D' is acyclic on N', and, thus, $N^0(N', D') \neq \varnothing$.

A partitioning $\mathcal{H} = \{H_1, \ldots, H_p\}$ of N can be generated through the following algorithm:

Step 1: Let $H_1 = N^0(N, D)$, $N_1 = N \setminus H_1$ and $D_1(i) = D(i) \cap N_1$ for all $i \in N_1$. Remark here that D_1 is an acyclic network on N_1.

Step $k+1$*:* Let $(H_1, \ldots, H_k)$ be generated in previous steps and let (N_k, D_k) be given as well.
If $N_k = \varnothing$, then terminate the algorithm and select $\mathcal{H} = (H_1, \ldots, H_p)$ where $p = k$.
If $N_k \neq \varnothing$ proceed as follows:

$$\begin{aligned} H_{k+1} &= N^0(N_k, D_k) \\ N_{k+1} &= N_k \setminus H_{k+1} \\ D_{k+1}(i) &= D(i) \cap N_{k+1} \text{ for every } i \in N_{k+1} \end{aligned}$$

Again remark here that D_{k+1} is an acyclic network on N_{k+1}. Then proceed to induction step $k+2$.

This algorithm terminates in a finite number of steps due to the finiteness of the player set N and the fact that $\#N_{k+1} < \#N_k$ in every step k. We thus arrive at a partitioning $\mathcal{H}$ of N as described, which clearly satisfies the requirements in the definition of a weakly hierarchical network.

Proof of Theorem 5.25

Proof of 5.25(a) First I introduce some auxiliary notions. Let $D \in \mathfrak{D}$ be a weakly hierarchical network. For D we define the corresponding *basic network benefit function* $b_D \colon \mathfrak{D} \to \mathbb{R}$ by

$$b_D(D') = \begin{cases} 1 & \text{if } D \subset D'; \\ 0 & \text{otherwise.} \end{cases} \tag{5.48}$$

The collection of basic network benefit functions $\{b_D \mid D \in \mathfrak{D}\}$ now forms a basis of the linear vector space of all network benefit functions on $\mathfrak{D}$, similar as the unanimity basis of the linear vector space of all cooperative games $\mathcal{G}^N$. (Shapley, 1953)

This leads to the conclusion that any network benefit function $r \colon \mathfrak{D} \to R$ can be decomposed into

$$r = \sum_{D \in \mathfrak{D}} \lambda_r(D)\, b_D \tag{5.49}$$

where $\{\lambda_r(D) \mid D \in \mathfrak{D}\}$ are unique determined by

$$\lambda_r(D) = \sum_{D' \subset D} (-1)^{|D|-|D'|} r(D'). \tag{5.50}$$

Using this, it follows from component additivity of r that

$$\sum_{C \in N/D} \sum_{D' \subset D_C} \lambda_r(D') = \sum_{C \in N/D} r(D_C) = r(D).$$

Using this and the fact that the decomposition of r into basic network benefit functions is unique, it can be shown that by using induction on the number of links that]$\lambda_r(D) = 0$ for all D with at least two components with at least two players, i.e., if there exist $C_1, C_2 \in N/D$ with $C_1 \neq C_2$ and $\min\{|C_1|, |C_2|\} \geqq 2$.

Furthermore, for $D \in \mathfrak{D}$ I introduce

$$\tau(D) = \{i \in N^0(N, D) \mid \text{There is some } j \in N \text{ with } j \in D(i)\} \tag{5.51}$$

as the collection of players that have successors but no predecessors in D. Since D is acyclic, we know that $\tau(D) \neq \varnothing$ if and only if $D \neq D_\varnothing$.

Now we are ready to construct an allocation rule γ on $\mathbb{S}(\mathfrak{D})$. For every $(N, r, D) \in \mathbb{S}(\mathfrak{D})$ and $i \in N$ define

$$\gamma_i(N, r, D) = \sum_{D' \subset D \colon i \in \tau(D')} \frac{\lambda_r(D')}{|\tau(D')|} \tag{5.52}$$

This allocation rule can be interpreted as follows: $\lambda_r(D')$ represents the contributions of the sub-network D'. Players in N divide this contribution among the "top players" in D' only, i.e., the players that have initiated links in D' without being a respondent in links in D'.

For γ we now claim that it satisfies CE as well as HPP on $\mathbb{S}(\mathfrak{D})$. Let $(N, r, D) \in \mathbb{S}(\mathfrak{D})$ and let $C \in N/D$ be a u-component of D. Then to show CE we derive that

$$\begin{aligned}\sum_{i\in C}\gamma_i(N,r,D) &= \sum_{i\in C}\sum_{D'\subset D:\, i\in\tau(D')}\frac{\lambda_r(D')}{|\tau(D')|}\\ &= \sum_{i\in C}\sum_{D'\subset D_C:\, i\in\tau(D')}\frac{\lambda_r(D')}{|\tau(D')|}\\ &= \sum_{D'\subset D_C:\, D'\neq\varnothing}\#\tau(D')\frac{\lambda_r(D')}{|\tau(D')|}\\ &= \sum_{D'\subset D_C:\, D'\neq\varnothing}\lambda_r(D') = r(D_C).\end{aligned}$$

To show HPP, remark that for any weakly hierarchical network $D \in \mathfrak{H}_w^N$ and $i, j \in N$ with $j \in D(i)$ we have that

$$\{D' \subset D \mid j \in \tau(D')\} = \{D' \subset (D - ij) \mid j \in \tau(D')\}.$$

The definition of γ now implies that for every $D \in \mathfrak{D}$ and $i, j \in N$ with $j \in D(i)$ it therefore holds that $\gamma_j(N, r, D) = \gamma_j(N, r, D-ij)$. Together with careful consideration of the definition of HPP, this means that γ indeed satisfies HPP.

Proof of 5.25(b) By Theorem 5.24, D^* contains a cycle $C = (i_1, \ldots, i_{m+1})$ with $i_1 = i_{m+1}$. Without loss of generality, assume that $t(C) = m$, i.e., $i_{k+1} \in D(i_k)$ for all $1 \leqq k \leqq m$. Define $D' \subset D_C \subset D$ by $D'(i_k) = \{i_{k+1}\}$ with $1 \leqq k \leqq m$.

Consider the benefit function r on $\mathfrak{D}$ given by

$$r(D) = \begin{cases} 1 & \text{if } D = D';\\ 0 & \text{otherwise.}\end{cases}$$

Suppose that γ is an allocation rule satisfying both CE as well as HPP on $\mathbb{S}(\mathfrak{D})$. Then with arguments used in Example 5.22 it follows that $\gamma_i(N, r, D'') = 0$ for all $D'' \subset D'$ and all $i \in N$. Furthermore, without loss of generality, we may assume that $\gamma_{i_1}(N, r, D') > 0$. Then HPP implies that for all $k \in \{1, \ldots, m\}$ it holds that

$$\gamma_{i_k}(N, r, D') = \alpha_k \cdot \gamma_{i_{k+1}}(N, r, D') \tag{5.53}$$

for some well chosen $\alpha_k \geqq 1$. From HPP and the hypothesis that $\gamma_{i_1}(N, r, D') > 0$, it now can be concluded that $\alpha_1 > 1$. Hence, with $i_{m+1} = i_1$:

$$\gamma_{i_1}(N, r, D') = \gamma_{i_{m+1}}(N, r, D') \cdot \prod_{k=1}^{m}\alpha_k > \gamma_{i_{m+1}}(N, r, D') = \gamma_{i_1}(N, r, D') > 0.$$

Thus, we have arrived at a contradiction.

Proof of Theorem 5.29

Some preliminary work is required to arrive at the proofs of the assertions stated in Theorem 5.29. We develop these preliminaries through three claims. Throughout we let $\alpha > 1$ and $\mathfrak{D} \subset \mathcal{D}^N$ be some closed collection of directed networks.

Claim 5.33 *There exists at most one allocation rule on $\mathbb{S}(\mathfrak{D})$ that satisfies both CE as well as α-HPP.*

Proof Suppose to the contrary that there exist two allocation rules γ^1 and γ^2 on $\mathbb{S}(\mathfrak{D})$ that satisfy both CE as well as α-HPP. We show that γ^1 coincides with γ^2 by induction on the number of links or relationships in the network.

First, let $(N, r, D) \in \mathbb{S}(\mathfrak{D})$ be such that there are no links, i.e., $D = D_\varnothing$. Then from CE it follows immediately that $\gamma_i^1(N, r, D) = \gamma_i^2(N, r, D) = r(D_\varnothing) = 0$ for all $i \in N$.

Second, let D consist of $R \geqq 1$ links and assume that γ^1 and γ^2 coincide on all sub-networks of D of at most $r - 1$ links. Let $i, j \in N$ with $j \in D(i)$. From α-HPP for γ^1 we then get that

$$\gamma_i^1(D) - \gamma_i^1(D - ij) = \alpha \left[\gamma_j^1(D) - \gamma_j^1(D - ij) \right]$$

Using this, the induction hypothesis and α-HPP of γ^2 we arrive at

$$\begin{aligned} \gamma_i^1(D) - \alpha\gamma_j^1(D) &= \gamma_i^1(D - ij) - \alpha\gamma_j^1(D - ij) \\ &= \gamma_i^2(D - ij) - \alpha\gamma_j^2(D - ij) = \gamma_i^2(D) - \alpha\gamma_j^2(D). \end{aligned}$$

Hence,

$$\gamma_i^1(D) - \gamma_i^2(D) = \alpha \left[\gamma_j^1(D) - \gamma_j^2(D) \right].$$

This expression is valid for all; pairs $i, j \in N$ with $j \in D(i)$.

Now let $i \in N$ and $C \in N/D$ be such that $i \in C$. For all $j \in C \setminus \{i\}$ there exists at least one u-path from i to j. Select exactly one such a path $P_{ij} = (i_1, \ldots, i_m)$ with $i_1 = i$ and $i_m = j$. For all $k \in \{1, \ldots, m-1\}$ it then holds that

$$\gamma_{i_k}^1(D) - \gamma_{i_k}^2(D) = \begin{cases} \alpha \left(\gamma_{i_{k+1}}^1(D) - \gamma_{i_{k+1}}^2(D) \right) & \text{if } i_{k+1} \in D(i_k) \\ \frac{1}{\alpha} \left(\gamma_{i_{k+1}}^1(D) - \gamma_{i_{k+1}}^2(D) \right) & \text{if } i_k \in D(i_{k+1}) \end{cases}$$

Hence, we can conclude that for all $j \in C$:

$$\gamma_i^1(D) - \gamma_i^2(D) = \alpha^{t(P_{ij})} \cdot \left(\gamma_j^1(D) - \gamma_j^2(D) \right).$$

Now define $w_j^C = \alpha(P_{ij})$ for all $j \in C \setminus \{i\}$ and $w_i^C = 1$. For all $j \in C$ it then follows that

$$\gamma_j^1(D) - \gamma_j^2(D) = w_j^C \cdot \left(\gamma_i^1(D) - \gamma_i^2(D)\right) = w_j^C \cdot d.$$

where $d = \gamma_i^1(D) - \gamma_i^2(D)$. Using CE for γ^1 and γ^2 we arrive at

$$\sum_{j \in C} \gamma_j^1(D) = \sum_{j \in C} \gamma_j^1(D) = r(D_C)$$

implying that

$$0 = \sum_{j \in C} \left(\gamma_j^1(D) - \gamma_j^2(D)\right) = \sum_{j \in C} w_j^C \, d.$$

Since $w_j^C > 0$ for all $j \in C$ it follows that $d = 0$. Since i was chosen arbitrarily we conclude that $\gamma^1(D) = \gamma^2(D)$. ■

For every hierarchical network $D \in \mathfrak{H}^N$ denote by $\mathcal{H}(D) = \{H_1, \ldots, H_p\}$ its hierarchical tier structure, i.e., for every $i, j \in N$ with $j \in D(i)$ there exists some k with $i \in H_{k+1}$ and $j \in H_k$. We note here that $\mathcal{H}(D)$ is unique if $\#N/D = 1$.

For the hierarchical network D we introduce also a weight vector $w_\alpha^D \in \mathbb{R}^N$ given by $(w_\alpha^D)_i = \alpha^k$ for all players $i \in H_k$, $k = 1, \ldots, p$. Thus, the weights of the players in D are based on the hierarchical position of that player in the tier system $\mathcal{H}(D)$ of the hierarchical network D. Hence, this weight system measures the "power" of a player in the hierarchy described by D.

Claim 5.34 *Let $D \in \mathfrak{H}^N$ be some hierarchical network and define $\mathfrak{D} = \{D' \mid D' \subset D\}$. Then the w_α^D-Network Myerson value satisfies CE as well as α-HPP on $\mathbb{S}(\mathfrak{D})$.*

Proof From Theorem 5.19 we conclude that the w_α^D-Network Myerson value $\mu^{w_\alpha^D}$ satisfies CE. It remains to be shown that $\mu^{w_\alpha^D}$ satisfies α-HPP. The quoted result, Theorem 5.19, states that $\mu^{w_\alpha^D}$ satisfies the w_α^D-balanced payoff property. Hence, for every $D' \subset D$ and all $i, j \in N$ with $j \in D'(i)$ it holds that

$$\frac{\mu_i^{w_\alpha^D}(D') - \mu_i^{w_\alpha^D}(D' - ij)}{\left(w_\alpha^D\right)_i} = \frac{\mu_j^{w_\alpha^D}(D') - \mu_j^{w_\alpha^D}(D' - ij)}{\left(w_\alpha^D\right)_j} \tag{5.54}$$

Since $\left(w_\alpha^D\right)_i = \alpha \left(w_\alpha^D\right)_j$, it holds that

$$\mu_i^{w_\alpha^D}(D') - \mu_i^{w_\alpha^D}(D' - ij) = \alpha \left(\mu_j^{w_\alpha^D}(D') - \mu_j^{w_\alpha^D}(D' - ij)\right) \tag{5.55}$$

Hence, the given value indeed satisfies α-HPP. ■

Claim 5.35 *Let $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathfrak{H}^N$ be two closed collections of hierarchical networks. If γ_1, respectively γ_2, satisfies CE as well as α-HPP on $\mathbb{S}(\mathfrak{D}_1)$, respectively $\mathbb{S}(\mathfrak{D}_2)$, then γ_1 and γ_2 coincide on $\mathbb{S}(\mathfrak{D}_1 \cap \mathfrak{D}_2)$.*

Proof Both γ^1 and γ^2 satisfy CE as well as α-HPP on $\mathbb{S}(\mathfrak{D}_1 \cap \mathfrak{D}_2)$. By the fact that $\mathfrak{D}_1 \cap \mathfrak{D}_2$ consists of hierarchical networks only, it follows from Claims 5.33 that $\gamma^1 = \gamma^2$ on $\mathbb{S}(\mathfrak{D}_1 \cap \mathfrak{D}_2)$. ■

Proof of 5.29(a) Define

$$\mathfrak{D}^{\max} = \{D \in \mathfrak{D} \mid \text{There is no } D' \in \mathfrak{D} \text{ with } D \subset D' \text{ and } D' \neq D\} \tag{5.56}$$

Let r be a network benefit function on $\mathfrak{D}$ and let $D \in \mathfrak{D}$. Select $D' \in \mathfrak{D}^{\max}$ with $D \subset D'$. Now introduce $\gamma(N, r, D) = \mu^{w_\alpha^{D'}}(N, r, D)$.

By Claim 5.34 it now follows that γ satisfies CE as well as α-HPP on $\mathbb{S}(\mathfrak{D}(D'))$ for any $D' \in \mathfrak{D}^{\max}$, where $\mathfrak{D}(D') = \{D \mid D \subset D'\}$. Claim 5.35 implies that $\gamma(N, r, D)$ is independent of the choice of $D' \in \mathfrak{D}^{\max}$. Thus, we conclude that γ satisfies CE and α-HPP on $\mathbb{S}(\mathfrak{D})$ as well. Claim 5.33 now guarantees that γ is the unique allocation rule on $\mathbb{S}(\mathfrak{D})$ that satisfies CE and α-HPP.

Proof of 5.29(b) Since D^* is non-hierarchical, it is not cycle regular either. So, let $C = \{i_1, \ldots, i_{m+1})$ be a cycle in D^* with $i_1 = i_{m+1}$ and $t(C) \neq 0$. As before let D' be the cyclic network based on C. Consider the network benefit function r on $\mathfrak{D}$ defined by

$$r(D) = \begin{cases} 1 & \text{if } D = D' \\ 0 & \text{otherwise} \end{cases} \tag{5.57}$$

Suppose that γ is an allocation rule on $\mathbb{S}(\mathfrak{D})$ satisfying CE and α-HPP. Then with arguments similar to the ones used in Example 5.22, it follows that $\gamma_i(N, r, D) = 0$ for all $D \subset D'$ and all $i \in N$. Subsequently, α-HPP implies that for all $k \in \{1, \ldots, m\}$ it holds that

$$\gamma_{i_{k+1}}(N, r, D') = \begin{cases} \alpha \gamma_{i_k}(N, r, D') & \text{if } i_k \in D'(i_{k+1}) \\ \frac{1}{\alpha} \gamma_{i_k}(N, r, D') & \text{otherwise.} \end{cases}$$

Hence, $\gamma_{i_{m+1}}(N, r, D') = \alpha^{-t(C)} \gamma_{i_1}(N, r, D')$. Now, since $t(C) \neq 0$ and $i_1 = i_{m+1}$ it follows that $\alpha = 1$ or $\gamma_{i_1}(N, r, D') = 0$. Because $\alpha > 1$, it has to hold that $\gamma_{i_1}(N, r, D') = 0$, which in turn implies that $\gamma_{i_k}(N, r, D') = 0$ for all $k \in \{1, \ldots, m\}$. This contradicts CE on D', since C itself is a component of D'.

5.5 Problems

Problem 5.1 Let $D \in \mathcal{D}^N$ be some directed network. Consider the corresponding inverse mapping D^{-1} on N. Show that the mapping D^{-1} contains exactly the same information as the original mapping D and that, in fact, the mappings D and D^{-1} are dual to one another.

Problem 5.2 This problem is based on the analysis of network power measures developed in van den Brink, Borm, Hendrickx, and Owen (2008). A directed network $D \in \mathcal{D}^N$ is called **binary** if it holds that

$$i \in D(j) \quad \text{if and only if} \quad j \in D(i).$$

The space of binary networks is denoted by $\mathcal{D}^N_*$.

For every binary network $D \in \mathcal{D}^N_*$ we define the corresponding **network power game** $p_D \in \mathcal{G}^N$ by

$$p_D(S) = \#\{j \in N \mid D^{-1}(j) \subset S\}.$$

Now consider the restriction of the β-measure on the space $\mathcal{D}^N_*$.

(a) Give the Harsanyi decomposition of the network power game p_D as a linear combination of unanimity basis games. In particular, formulate the Harsanyi dividends $\Delta_{p_D}(S)$, $S \subset N$, of the game p_D.
(b) Show that the β-measure of D is exactly the Shapley value of the corresponding network power game, i.e., $\beta(D) = \varphi(p_D)$.
(c) Show that a measure $p \colon \mathcal{D}^N_* \to \mathbb{R}^N$ is equal to the β-measure on $\mathcal{D}^N_*$ if and only if p satisfies the following six axioms:

1. *Dominance normalization:* $\sum_{i \in N} p_i(D) = \#\{j \in N \mid D(j) \neq \varnothing\}$.
2. *Symmetry:* For every permutation $\pi : N \leftrightarrows N$ it holds that $p_i(D) = p_{\pi(i)}(\pi D)$, where πD is defined by $\pi(j) \in \pi D(\pi(i))$ if and only if $j \in D(i)$.
3. *Reasonability:* For every $i \in N$ it holds that

$$\#\{j \in D(i) \mid D(j) = \{i\}\} \leqq p_i(D) \leqq \#D(i).$$

4. *Non-neighborhood independence:* Let $D, D' \in \mathcal{D}^N_*$ and $i \in N$ be such that $D(j) = D'(j)$ for all $j \in D(i) \cup \{i\}$. Then $p_i(D) = p_i(D')$.
5. *Link cutting independence:* Let $D \in \mathcal{D}^N_*$ be such that $i \in D(j)$ and $D(h) = D(k) = \varnothing$. Construct $D' \in \mathcal{D}^N_*$ such that $D'(i) = D(i) - j + h$, $D'(j) = D(j) - i + k$, $D'(h) = \{i\}$ and $D'(k) = \{j\}$. Then $p_\iota(D) = p_\iota(D')$ for every $\iota \in N \setminus \{i, j\}$.
6. *Pending position addition:* Let $D \in \mathcal{D}^N_*$ be such that $D(j) = \varnothing$. Define $D' \in \mathcal{D}^N_*$ by $D'(i) = D(i) + j$, $D'(j) = \{i\}$ and $D'(h) = D(h)$ for all $h \neq i, j$. Then

$$p_h(D) - p_h(D') = p_k(D) - p_k(D')$$

for all $h, k \in D(i)$.

(d) Using the formulation of the proper Shapley value stated in (3.49), compute the regular Shapley value as well as the proper Shapley value for p_D if the network D is a line-network. (I refer here to Hendrickx, Borm, van den Brink, and Owen (2009) for a discussion of the application of the proper Shapley value to the game p_D.)

Problem 5.3 The Copeland score measure is given by the measure $C\colon \mathcal{D}^N \to \mathbb{R}^N$ defined by

$$C_i(D) = \#D(i) - \#D^{-1}(i). \tag{5.58}$$

Determine which properties the Copeland score measure and the degree measure have in common. Give some examples of directed networks for which the various measures such as the degree, the β, the modified β, the Copeland score, and the positional power measures, result into different evaluations of the various positions in the directed network.

Problem 5.4 Construct a proof of Lemma 5.16.

Problem 5.5 Prove the equivalence of statements (i) and (ii) in Theorem 5.24. Refer to Definition 2.30 for the precise definition of a strict permission structure.

Problem 5.6 Let (N, r, D) be some cooperative network situation for the closed collection $\mathfrak{D} = \{D' \mid D'(i) \subset D(i) \text{ for all } i \in N\}$. Let the β-Network Myerson value μ^β be determined as the $\beta(D)$-Network Myerson value on $\mathfrak{D}$, where $\beta(D)$ is the β-measure for network D.[11]

(a) Show that $\mu^\beta(N, r, D)$ is well defined for all directed networks $D \in \mathcal{D}^N$ and network benefit functions r.
(b) Determine whether the β-Network Myerson value satisfies the hierarchical payoff property. If not, construct a counter example.

[11] Note that the β-Network Myerson value is fundamentally different from an arbitrary w-Network Myerson value, since the weights are not fixed, but determined by the corresponding β-measure of D.

Chapter 6
Cooperative Theories of Hierarchical Organizations

The goal of this chapter is to analyze the consequences of the implementation of a hierarchical authority structure on the set of players in the context of a cooperative game with transferable utilities. In this analysis it is assumed that an exogenously given authority structure is imposed and puts certain constraints on the behavior of the players in the game. I consider the consequences of such constraints in the form of restricted games that incorporate these restrictions. This results into modifications of the standard Myerson values for such games. In the subsequent presentation I mainly focus on the properties and axiomatizations of these generated values for such hierarchically structured games. This chapter draws heavily from Gilles et al. (1992) and Gilles and Owen (1999).

In the previous chapter I already debated the natural properties of domination networks as well as information transfer networks. The main conclusion for such information networks was that hierarchical reward systems could only be implemented in hierarchically structured organizations. In the current chapter the analysis is carried one step further and I consider situations in which such hierarchical structures are imposed exogenously. As such, the main motivation to analyze the implementation of such a hierarchical structure on a cooperative game is that it is a natural representation of *authority*. I refer to Chapter 5 for further details.

As discussed in the previous chapter, this naturally leads to the consideration of a certain subclass of authority networks denoted as *permission structures*. Within these networks, there is an explicit implementation of authority in the form of full control of a subordinate's actions by a superior. Here, it is assumed that there is a production process founded on a proprietary production technology. A superior has the authority to limit the access of a subordinate to this production technology. This is usually modeled as that the superior has a certain *veto power* over the subordinate's access to the production technology. In this regard, the subordinate is not able to generate certain added values using such technology.

In this chapter the analysis considers an abstract hierarchical authority structure in which certain players have well specified veto power over the activities undertaken by certain other players. A controlling player is denoted as a *superior* and the partially controlled player, subject to the superior's veto power, is denoted as a *subordinate* of that particular superior. The consequences of these specified forms

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of veto-based constraints on any cooperative game are then studied using the general analytical and computational framework seminally outlined by Owen (1986).[1]

To illustrate the motivation of the use of hierarchical structures to represent such veto power situations I use a simple production situation with three traders, two equally powerful owners of a production technology and one worker or provider of human capital, who can extract one unit of output by combining his human capital with the given production technology:

Example 6.1 I consider a production situation with three players, described by the set $N = \{1, 2, 3\}$. Suppose that only player 1 is productive and creates an output of one unit using a given production technology. If the production technology is freely available, this situation can be described by a the game (N, v) given by $v = u_{\{1\}}$, i.e., $v(S) = 1$ if $1 \in S \subset N$, and $v(S) = 0$ otherwise.

Now suppose that players 2 and 3 own the given production technology. Naturally this implies that both players 2 and 3 control the access of this production technology. As such this can be represented by a hierarchical authority structure in which players 2 and 3 are both superiors of player 1 as their subordinate. What are the consequences of this hierarchical authority structure for the productivity as well as the rewards of the three decision makers involved?

There are several possibilities to specify the power of the superiors in the authority situation over their subordinates. In the *conjunctive approach*—seminally developed in Gilles et al. (1992)—one assumes that every player has unequivocal full veto power over the actions undertaken by all of her subordinates. In that case, player 1 cannot produce his one unit of output without the full cooperation of both his superiors. This is then represented by $w^c = u_N$, i.e., $w^c(N) = 1$ and $w^c(S) = 0$ for $S \neq N$. The resulting game w^c is called the *conjunctive restriction* of the game v with regard to the hierarchical authority structure described.

In the seminal paper Gilles and Owen (1999) the conjunctive hypothesis is modified to the other standard case in which a subordinate only has to get permission from *at least one* superior within the hierarchical authority structure. Here the power to veto is more limited and turned into a form of authorization of access to the production technology; any superior is assumed to be able to grant such authorization. Thus, an action of a certain player has to be authorized by a chain of subsequent superiors within the hierarchy. To distinguish this approach from the conjunctive approach we denote this behavioral model as the *disjunctive approach*.[2]

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[1] Owen (1986) provides an analysis of cooperative games with limited communication. His method can however be transferred easily to the analysis of games with authority structures as is the case developed in Gilles et al. (1992) and Gilles and Owen (1999). I also point out that a different approach to this subject originated with the work of Faigle (1989), Faigle and Kern (1993), and Derks and Peters (1993). These contributions to the literature, however, use a more general approach based on purely mathematical considerations of generalizing the Shapley value to arbitrary cooperation structures rather than the explicit study of hierarchical authority structures as pursued in Gilles et al. (1992).

[2] For a third, alternative approach I also refer to Winter (1989).

Let us consider again the three-player example of the hierarchical production situation introduced above. In contrast to the conjunctive approach, within a disjunctive hierarchy player 1 has to obtain permission to produce from either player 2 or player 3 or both. Such a production situation can be represented by a game $w^d \in \mathcal{G}^N$ with $w^d(S) = 1$ if $S \in \{12, 13, 123\}$ and $w^d(S) = 0$ otherwise. The game w^d is referred to as the *disjunctive restriction* of the original production game v. ■

In general we denote a situation with a hierarchical authority structure as a *cooperative game with a permission structure,* consisting of a set of players, a cooperative game with transferable utilities describing the potential outputs of the various coalitions, and a mapping that assigns to every player a subset of (direct) subordinates describing the authority structure. Now a coalition can only form if every member is authorized to participate and thus for each member either all of his superiors (conjunctive) or at least one superior are members of that coalition. This leads to a reduced collection of formable coalitions. In other words, a coalition can only generate its potential output if it is *autonomous* within the permission structure. By adopting that only autonomous coalitions generate their potential output, we arrive at the desired conjunctive or disjunctive restriction of the original cooperative game. This is exactly what is developed in the three player example above.

In this chapter I discuss in length the relationship of original production games and their conjunctive and disjunctive restrictions for arbitrary hierarchical authority situations. Also I show how the Shapley value is affected in such situations. This is pursued by considering the Shapley value of the restricted games directly, rather than considering the Myerson value based on the collection of autonomous coalitions.

6.1 Games with a Permission Structure

Authority already has been subject of discussion in Chapter 2, in particular Definition 2.30 and the subsequent discussion. There I limited the discussion mainly to the so-called conjunctive permission structures in which authority is exercised in a precisely defined fashion. Here I return to this subject and enhance the definition that we used in Chapter 2.

I emphasize that the notation used in this chapter is developed independently from the notation employed in the previous chapter on directed networks. I note, however, that the notation used is rather similar.

Definition 6.2 A map $H{:}N \to 2^N$ is a *permission structure* on player set N if it is irreflexive, i.e., $i \notin H(i)$ for all $i \in N$.

The players $j \in H(i)$ are called the *direct subordinates* of player i in H. The players $j \in H^{-1}(i)$ with

$$H^{-1}(i) = \{j \in N \mid i \in H(j)\}$$

are called the *direct superiors* of player i in H.
The class of all permission structures on N is denote as $\mathcal{H}^N$.

Of course, permission structures are formally defined in exactly the same fashion as directed networks, explored in Chapter 5. In fact, mathematically $\mathcal{H}^N = \mathcal{D}^N$. However, there is a very significant difference; in such directed networks the relationships are considered primarily to be communication links, while in permission structures the relationships describe authority relationships. Hence, I explicitly assume in the subsequent analysis in this chapter that $j \in H(i)$ implies that player i has direct authority over player j with regard to his productive activities. We arrive at a structure that has similar mathematical properties as the ones explored in Chapter 5, but here these structures and their properties have very different interpretations.

The interpretation of an authority relationship in $H \in \mathcal{H}^N$ leads to different insights than the interpretation of a directed communication relationship in a communication network $D \in \mathcal{D}^N$. In fact, the analysis will be very different. We already explored the consequences of the hierarchical payoff property in directed communication networks; in this chapter the identified hierarchical structures will be our starting point in the modeling of hierarchical authority structures or "hierarchies" in production organizations such as firms.

A (directed) *authority path* from player i to player j in permission structure H is a finite sequence of players $P = (i_1, \dots, i_m)$ such that $i_1 = i$, $i_m = j$ and $i_{k+1} \in H(i_k)$ for all $k \in \{1, \dots, m-1\}$. A path $P = (i_1, \dots, i_m)$ from a player i to herself in H is denoted as an authority *cycle*, i.e., $i_1 = i_m = i$.

The notion of a path allows the introduction of the transitive closure of the permission structure H. Similarly as discussed in Chapter 5, the mapping $H^+ : N \to 2^N$ is the *transitive closure* of permission structure H if for every player $i \in N$:

$$H^+(i) = \{j \in N \mid \text{There exists a path from } i \text{ to } j \text{ in } H\}. \tag{6.1}$$

Similarly, the mapping $H^- : N \to 2^N$ is the *inverse* of the transitive closure of permission structure H if for every player $i \in N$:

$$H^-(i) = \{j \in N \mid \text{There exists a path from } j \text{ to } i \text{ in } H\}. \tag{6.2}$$

Obviously it holds that

$$j \in H^+(i) \text{ if and only if } i \in H^-(j).$$

A player $j \in H^+(i)$ is denoted as an (indirect) *subordinate* of player i within the authority structure H. Similarly, player i is the (indirect) *superior* of player j. In our subsequent discussions, I refer to players $j \in H^+(i)$ plainly as the subordinates of player i in H and players $j \in H^-(j)$ as superiors of player i. I mention that since (i) is a path from i to herself, it holds that $i \in H^-(i)$ as well as $i \in H^+(i)$, i.e., every player i is a subordinate as well as a superior of herself. This property should be interpreted as a technicality within the mathematical setting of a permission structure.

Definition 6.3 Let $H{:}N \to 2^N$ be a permission structure on N.

(i) A permission structure $H{:}N \to 2^N$ is *weakly hierarchical* or, more simply, ***strict*** if H is acyclic.
The class of strict permission structures on N is denoted by $\mathfrak{H}^N_w \subset \mathcal{H}^N$.
(ii) A permission structure $H{:}N \to 2^N$ is *strictly hierarchical* if H is ***strict*** and there exists a unique player $i_H \in N$ without superiors, i.e., $H^-(i_H) = \varnothing$ and $H^-(j) \neq \varnothing$ for all $j \neq i$.
The class of strictly hierarchical permission structures on N is denoted by $\mathfrak{H}^N_s \subset \mathfrak{H}^N_w$.

With reference to the discussion of hierarchical networks in Chapter 5, I remark that weakly hierarchical permission structures are the same as weakly hierarchical or acyclic directed networks. This justifies the chosen notation of the class of weakly hierarchical permission structures.

On the other hand, strictly hierarchical permission structures form a different subclass of the class of weakly hierarchical permission structures than the hierarchical networks introduced in Chapter 5.[3]

In our subsequent discussion the standard assumption is that a permission structure is at least weakly hierarchical and in many cases strictly hierarchical. Also, for any strictly hierarchical permission structure $H \in \mathfrak{H}^N_s$ I denote by $i_H \in N$ the unique player without any superiors in H. The proof of the property stated in Lemma 6.4 is left as an exercise.

Lemma 6.4 *For any strictly hierarchical permission structure $H \in \mathfrak{H}^N_s$ it holds that $H^+(i_H) = N$ and i_H is the unique player $i \in N$ with $H^+(i) = N$.*

The assertion stated in Lemma 6.4 clarifies that in a strict hierarchy there is a unique position of an overall superior, indicated by i_H. This player i_H is the unique player who is the superior of any other player in the permission structure. In some strands of the literature on cooperative games, such a powerful individual is also denoted as the "big boss". Here I will use the term *executive player* to indicate the individual i_H that occupies such a position of ultimate authority in the authority structure represented by H.

The definitions introduced thus far can be expanded straightforwardly to coalitions of players. For any coalition $S \subset N$ I can introduce the following concepts: $H(S) = \cup_{i \in S} H(i)$, $H^{-1}(S) = \cup_{i \in S} H^{-1}(i)$, $H^-(S) = \cup_{i \in S} H^-(i)$ and $H^+(S) = \cup_{i \in S} H^+(i)$ stand for the direct subordinates, the direct superiors, all subordinates and all superiors of coalition S, respectively.

The introduction of permission structures allows us to introduce the main concept of this chapter, namely a cooperative game endowed with an exogenously given permission structure, representing the authority relations between the various players.

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[3] In fact, it holds that $\mathfrak{H}^N_s \neq \mathfrak{H}^N$ with the properties that $\mathfrak{H}^N_s \cap \mathfrak{H}^N \neq \varnothing$, $\mathfrak{H}^N_s \setminus \mathfrak{H}^N \neq \varnothing$, $\mathfrak{H}^N \setminus \mathfrak{H}^N_s \neq \varnothing$ and $\mathfrak{H}^N_s \cup \mathfrak{H}^N \subsetneq \mathfrak{H}^N_w$.

From an alternative perspective, the cooperative game represents the output levels of the various coalitions if they had access to certain assets that are under the control of the players in the authority structure.

Definition 6.5 *A game with a permission structure* is a triple (N, v, H) where $N = \{1, \dots, n\}$ is a finite player set, $v \in \mathcal{G}^N$ is a cooperative game on N, and $H \in \mathcal{H}^N$ is a permission structure on N.

The space of all games with a permission structure on a given player set N is now simply given by $\mathcal{G}^N \times \mathcal{H}^N$.

Hence, a game with a permission structure summarizes the data of a hierarchical production organization. The player set N denotes the members of the organization, the cooperative game v summarizes the productive abilities of these members, and the permission structure H describes the exogenously given authority relationships between the various members of the organization.

What is lacking from the formal definition of a game with a permission structure is the specification how authority is exercised. Thus, the theory has to be extended to capture how veto power is actually implemented. In the introduction I already discussed the fundamentals of the two main decision rules of exercising authority that we discuss here. These two fundamental approaches are formalized and discussed further in the two subsections below.

6.1.1 The Conjunctive Approach

As stated before, coalition formation within the context of a permission structure can be described in two fundamentally different ways. The conjunctive approach has seminally been developed in Gilles et al. (1992). The main hypothesis of the exercise of authority can be formalized as follows:

Definition 6.6 Let H be a permission structure on N. A coalition $S \subset N$ is said to be *conjunctively autonomous* within H if $S \cap H(N \setminus S) = \varnothing$. The collection of conjunctively autonomous coalitions is denoted by

$$\Gamma_H = \{S \subset N \mid S \cap H(N \setminus S) = \varnothing\}. \tag{6.3}$$

In general, as stated, a coalition is "autonomous" if it contains all of the required superiors for the coalition to operate. Thus, every member is authorized to participate in the value-generating process that this coalition can engage in. In the conjunctive setting this means that there is no player that has any indirect authority over any member in the coalition, who is not member of that coalition.

Following the discussion in Chapter 2, the collection Γ_H of conjunctively autonomous coalitions forms a lattice on N and, moreover, Γ_H is a discerning lattice if and only if H is a weakly hierarchical permission structure. Finally, if H is a strictly hierarchical permission structure, then

$$\cap \{S \in \Gamma_H \mid S \neq \varnothing\} = \{i_H\}. \tag{6.4}$$

Hence, the executive player i_H is a member of *every* conjunctively autonomous coalition in the strictly hierarchical permission structure H. Again, the executive player i_H is the unique player with that property.

With the lattice of conjunctively autonomous coalitions I can now introduce several important related coalitions that describe the relationship of a coalition with certain other players in the permission structure H. Let $S \subset N$ be some coalition and let $H \in \mathcal{H}^N$. Then I introduce

$$\gamma_H(S) = \cup \{T \in \Gamma_H \mid T \subset S\} \equiv \{i \in S \mid H^{-1}(i) \subset S\} \equiv S \setminus H^+(N \setminus S) \qquad (6.5)$$

as the *conjunctively autonomous part* of coalition S. In fact, the autonomous part of a coalition is its maximal autonomous subcoalition. This subcoalition consists exactly of all players in S that have no superiors outside S. These are indeed the players that in the conjunctive setting are not vetoed and have unhampered access to the required assets to generate an output. In fact, they can generate the output level $v(\gamma_H(S))$.

Furthermore, I introduce

$$\gamma_H^-(S) = \cap \{T \in \Gamma_H \mid S \subset T\} \equiv \{i \in N \mid H^+(i) \cap S \neq \varnothing\} \equiv H^-(S) \qquad (6.6)$$

is denoted as the *conjunctively authorizing cover* of coalition S. The authorizing cover consists exactly of those players that have a subordinate who is a member of the coalition S. Indeed, in the conjunctive approach these are exactly the players that are required to give permission—in the sense of abstaining from the exercise of their right to veto—for coalition S to get access to the required assets to generate the output $v(S)$.[4]

Without loss of generality I will use the more general notation $\gamma(S)$ for the autonomous part of coalition S in permission structure H if the permission structure H is unambiguous. Similarly, I will use $\gamma^-(S)$ for the authorizing set of coalition S if its use is unambiguous.

We can now state some basic properties of the autonomous part of a coalition as well as its authorizing cover. For a proof of Proposition 6.7 I refer as usual to the appendix of this chapter.

Proposition 6.7 *Let $H \in \mathcal{H}^N$ be some permission structure and let $S, T \subset N$ be some coalitions. Then the following properties hold:*

(i) $\left[\gamma_H(S) \cup \gamma_H(T)\right] \subset \gamma_H(S \cup T)$;
(ii) $\gamma_H(S) \cap \gamma_H(T) = \gamma_H(S \cap T)$;
(iii) $\gamma_H^-(S) \cup \gamma_H^-(T) = \gamma_H^-(S \cup T)$, *and*
(iv) $\gamma_H^-(S \cap T) \subset \left[\gamma_H^-(S) \cap \gamma_H^-(T)\right]$.

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[4] It should be clear that in the conjunctive approach, the players in S simply do not have access to the required assets to be productive without the full participation of their superiors. Thus, coalition S is powerless, while its authorizing set $\gamma_H^-(S)$ is the smallest supercoalition that would provide the necessary access to these productive assets.

The basic idea underlying the conjunctive approach is now that in a game with a permission structure (N, v, H) a coalition $S \subset N$ cannot have full access to the productive assets to generate $v(S)$ without the full participating of *all* superiors of the members of S. This implies that within S only the members of its autonomous part $\gamma(S)$ have access to these assets. This results into the following concept:

Definition 6.8 Let $(N, v, H) \in \mathcal{G}^N \times \mathcal{H}^N$ be a game with a permission structure. The *conjunctive restriction* of (N, v, H) is the game $\mathfrak{R}_H(v) \in \mathcal{G}^N$ defined by

$$\mathfrak{R}_H(v)(S) = v\left(\gamma_H(S)\right) \qquad \text{for all } S \subset N \tag{6.7}$$

The conjunctive restriction for a given permission structure $H \in \mathcal{H}^N$ can now be understood as a mapping $\mathfrak{R}_H: \mathcal{G}^N \to \mathcal{G}^N$ from the space of cooperative games on N into itself. Next I state the main mathematical properties of this mapping.

Theorem 6.9 *Let $H \in \mathcal{H}^N$ be some permission structure. The conjunctive restriction $\mathfrak{R}_H$ for H is a linear projection mapping of rank $\#\Gamma_H - 1$ on $\mathcal{G}^N$, where $\#\Gamma_H - 1$ is the number of nonempty conjunctively autonomous coalitions in H. Its kernel (or null space) is spanned by the standard basis games $\{b_S \mid S \notin \Gamma_H\}$ and its image is spanned by the unanimity basis games $\{u_S \mid S \in \Gamma_H\}$.*

From the above it is clear that the conjunctive restriction is a mapping with some very powerful mathematical properties. The next result gives an exact formulation of this mapping. The proof of Theorem 6.10 is combined with the proof of Theorem 6.9 and relegated to the appendix of this chapter.

Theorem 6.10 *Let (N, v, H) be a game with a permission structure. Then its conjunctive restriction is characterized by*

$$\mathfrak{R}_H(v) = \sum_{S \in \Gamma_H} \left(\sum_{T \subset N : \gamma_H^-(T) = S} \Delta_v(T) \right) \cdot u_S. \tag{6.8}$$

Next we extend our point of view and consider the conjunctive restriction to be a mapping on the space of all games with a permission structure. Formally, the conjunctive restriction can be considered as a mapping $\mathfrak{R}: \mathcal{G}^N \times \mathcal{H}^N \to \mathcal{G}^N$ given by $\mathfrak{R}(v, H) = \mathfrak{R}_H(v)$. This mapping has certain specific properties that completely characterize it. The following axiomatization of the conjunctive restriction $\mathfrak{R}$ was developed by van den Brink and Gilles (2009). That paper goes on to use the conjunctive restriction to describe the consequences of the exercise of authority in hierarchical authority structures. Here I limit myself to the presentation of this illuminating characterization.

Theorem 6.11 *A mapping $\mathfrak{F}: \mathcal{G}^N \times \mathcal{H}^N \to \mathcal{G}^N$ is equal to the conjunctive restriction $\mathfrak{R}$ if and only if the mapping $\mathfrak{F}$ satisfies the following five properties:*

(i) *For every $(v, H) \in \mathcal{G}^N \times \mathcal{H}^N$ it holds that $\mathfrak{F}(v, H)(N) = v(N)$;*

(ii) *For every* $(v,H),(w,H) \in \mathcal{G}^N \times \mathcal{H}^N$ *it holds that* $\mathfrak{F}(v+w,H) = \mathfrak{F}(v,H) + \mathfrak{F}(w,H)$*;*
(iii) *For every* $(v,H) \in \mathcal{G}^N \times \mathcal{H}^N$ *and* $i \in N$ *such that all* $j \in H^+(i) \cup \{i\}$ *are null players in* v *it holds that* i *is a null player in* $\mathfrak{F}(v,H)$*;*
(iv) *For every* $(v,H) \in \mathcal{G}^N \times \mathcal{H}^N$ *and* $i \in N$ *such that* $v(S) = 0$ *for all coalitions* $S \subset N \setminus \{i\}$ *it holds that* $\mathfrak{F}(v,H)(S) = 0$ *for all* $S \subset N \setminus \{i\}$*;*
(v) *For every* $(v,H) \in \mathcal{G}^N \times \mathcal{H}^N$*,* $i \in N$*,* $j \in H(i)$*, and* $S \subset N \setminus \{i\}$ *it holds that* $\mathfrak{F}(v,H)(S) = \mathfrak{F}(v,H)(S \setminus \{j\})$*.*

Next I consider two examples of applications of the conjunctive approach to productive situations with hierarchical authority structures. First, I discuss the natural case of an additive production technology.

Example 6.12 Consider a hierarchical authority structure represented by a strict permission structure $H \in \mathcal{H}^N$ on a player set N. Let $\lambda N \to \mathbb{R}_{++}$ be a strictly positive weight system. Now we define the additive game based on λ by

$$v_\lambda(S) = \sum_{i \in S} \lambda_i \quad \text{for all } S \subset N. \tag{6.9}$$

Hence, every player $i \in N$ is a productive worker who generates an added value of $\lambda_i > 0$. The superiors of player i all have to give permission to that player before he can generate that added value. This is exactly represented by the conjunctive restriction $w_\lambda = \mathfrak{R}_H(v_\lambda)$.

It is easy to compute that the Harsanyi dividends of the conjunctive restriction w_λ are given by

$$\Delta_{w_\lambda}(S) = \sum_{i \in N : H^-(i) = S} \lambda_i \tag{6.10}$$

where $S \subset N$ with $S \neq \varnothing$.

Furthermore, since H is weakly hierarchical, it cannot hold that $i \in H^+(j)$ as well as $j \in H^+(i)$. Therefore, $H^-(i) \neq H^-(j)$ for all $i \neq j$. This implies that

$$\Delta_{w_\lambda}(S) = \begin{cases} \lambda_i & \text{if } S = H^-(i) \text{ for some } i \in N \\ 0 & \text{otherwise.} \end{cases} \tag{6.11}$$

This implies that we can write

$$w_\lambda = \sum_{i \in N} \lambda_i \cdot u_{H^-(i)}. \tag{6.12}$$

With the computed Harsanyi dividends we can easily determine the corresponding Shapley values. First, let $\sigma_H(i) = \# H^-(i) \geqq 1$ be the number of superiors of player i in the hierarchical authority structure H (including himself). Then:

$$\varphi_i(w_\lambda) = \sum_{j \in N : i \in H^-(j)} \frac{\lambda_j}{\sigma_H(j)} = \sum_{j \in H^+(i)} \frac{\lambda_j}{\sigma_H(j)} \tag{6.13}$$

This in principle defines another power measure on a directed network. In particular for $\lambda_i = 1$ for all $i \in N$ we have derived the β-measure of the directed network represented by the hierarchy H discussed in the previous chapter. ■

I conclude the discussion of the conjunctive approach to games with a permission structure with a fundamental example of the role of the executive in comparison with middle management in a production organization.

Example 6.13 I consider a very specific hierarchical production organization. Let $N = \{p, w\} \cup M$ where $M = \{1, \ldots, m\}$. On N we define the game $v = u_{\{w\}}$ as the unanimity game for $\{w\}$. Finally, H is given by $H(p) = M$, $H(j) = \{w\}$ for $j \in M$, and $H(w) = \varnothing$.

The triple (N, v, H) now represents a simple production hierarchy with middle management and a single productive worker. Here p represents the executive player or "CEO" of the production organization, w is the unique worker who generates a unit output, and M is a set of middle managers.

With our previous results it is easy to compute that

$$v' = \mathfrak{R}_H(v) = u_N$$

and that the Shapley value for this production hierarchy is completely egalitarian and given by $\varphi_i(v') = \frac{1}{m+2}$. In the discussion of the disjunctive approach in the next section I will consider this game again and determine that the resulting division of the organization's output is completely different. ■

6.1.2 The Disjunctive Approach

Next I consider the formation of coalitions under what is indicated as the disjunctive approach to games with a permission structure. In the disjunctive approach it is assumed that each player has to receive permission of *at least one* direct superior to attain access to the productive assets of the organization as opposed to permission of *all* of his superiors in the conjunctive approach.

The disjunctive approach has been introduced seminally in an early draft of Gilles and Owen (1999) and has been further developed by van den Brink (1997). In this section I present the main results and discussions from Gilles and Owen (1999).

For technical reasons, we have to restrict ourselves to weakly hierarchical or strict permission structures $H \in \mathfrak{H}^N_w$. In such a strict permission structure H we define

$$B_H = \{i \in N \mid H^{-1}(i) = \varnothing\} \tag{6.14}$$

as the set of "executive players". These players are exactly those who have no superiors in the permission structure H. Since H is weakly hierarchical, it is clear that $B_H \neq \varnothing$. Moreover, if the permission structure H is strictly hierarchical, then obviously $B_H = \{i_h\}$ consists of the unique principal executive player already introduced in the previous discussion.

As in the discussion of the conjunctive approach I first address which coalitions of players in (N, v, H) can form in the sense of getting full permission to realize the potential productive value assigned in the game v. These are exactly those coalitions which members have at least one direct superior or predecessor in that coalition:

Definition 6.14 A coalition $S \subset N$ is *disjunctively autonomous* in the strict permission structure $H \in \mathfrak{H}_w^N$ if for every $i \in S \setminus B_H$:$H^{-1}(i) \cap S \neq \varnothing$.

The collection of disjunctively autonomous coalitions in the weakly hierarchical permission structure H is indicated by Ψ_H.

From the definition it follows that a coalition $S \subset N$ is disjunctive autonomous in H if and only if for every player $i \in S$ there exists a collection of players $\{j_1, \ldots, j_m\} \subset S$ with $j_1 \in B_H$, $j_m = i$, and for every $1 \leqq k \leqq m - 1$ it holds that $j_{k+1} \in H(j_k)$.

It should be clear that the collection of disjunctively autonomous coalitions Ψ_H has a more complicated structure than the collection of conjunctive autonomous coalitions Γ_H. With regard to the collection Ψ_H we can prove only that it is closed for taking unions:

Proposition 6.15 *Let $H \in \mathfrak{H}_w^N$ be a strict permission structure. Then $\varnothing, N \in \Psi_H$ and for every $S, T \in \Psi_H$:$S \cup T \in \Psi_H$.*

Proof Evidently $\varnothing \in \Psi_H$ as well as $N \in \Psi_H$. Take $S, T \in \Psi_H$ and $i \in S \setminus B_H$. Then by definition $\varnothing \neq H^{-1}(i) \cap S \subset H^{-1}(i) \cap [S \cup T]$. Similarly for $i \in T \setminus B_H$. This shows indeed that $S \cup T \in \Psi_H$. ■

In Example 6.16 below we show that Ψ_H does not have to be closed for taking intersections. The knowledge that finite unions of disjunctively autonomous coalitions are again disjunctively autonomous leads us to the introduction of the maximal disjunctively autonomous subcoalition of any given coalition in the setting of an acyclic permission structure.

Let $H \in \mathfrak{H}_w^N$ be a strict permission structure on N and let $S \subset N$. Then the subcoalition given by

$$\delta_H(S) := \cup\{T \in \Psi_H \mid T \subset S\} \tag{6.15}$$

is the *disjunctively autonomous part* of E in S. Again the autonomous part of coalition S is simply the maximal disjunctive autonomous subcoalition of S.

Because of the limited properties of the collection of disjunctively autonomous coalitions, there cannot be identified a unique disjunctive autonomous cover of the coalition S. A coalition $T \subset N$ is a *disjunctive authorizing cover* of S in the strict permission structure H if

(i) $T \in \Psi_H$ and $S \subset T$, and
(ii) there does not exist a $U \in \Psi_H$ such that $S \subset U \subset T$ and $U \neq T$.

The collection of all authorizing covers of S in H is denoted by $\mathfrak{A}_H(S) \subset \Psi_S$.

Clearly, $S \in \Psi_H$ if and only if for every member $i \in S$ there is a disjunctive authorizing set $T_i \in \mathfrak{A}_H(\{i\})$ with $T_i \subset S$.

Using this, another characterization of disjunctively autonomous coalitions based on authorizing covers is given by

$$\Psi_H = \{S \subset N \mid \mathfrak{A}_H(S) = \{S\}\} \quad \text{and} \quad \Psi_H = \bigcup_{S \subset N} \mathfrak{A}_H(S).$$

Furthermore, I mention without proof that for every non-empty coalition $\varnothing \neq S \subset N$ it holds that $\mathfrak{A}_H(s) \neq \varnothing$.

As a special case I consider the "empty" permission structure H_0 defined by $H_0(i) = \varnothing$ for every $i \in N$. Obviously, $\Psi_{H_0} = 2^N$ and for every $S \subset N$ we have $\mathfrak{A}_{H_0}(S) = \{S\}$ and $\delta_{H_0}(S) = S$. Later I use the empty permission structure to show that the disjunctive approach indeed generalizes the standard approach to cooperative games.

I conclude this section with an example to illustrate these concepts.

Example 6.16 Take $N = \{1, 2, 3, 4, 5\}$. Consider the permission structure H by $H(1) = \{2, 3\}$, $H(2) = \{4\}$, $H(3) = \{4, 5\}$, and $H(4) = H(5) = \varnothing$. We can represent this strictly hierarchical permission structure by a directed graph on the set of players N. This graph is given in Fig. 6.1.

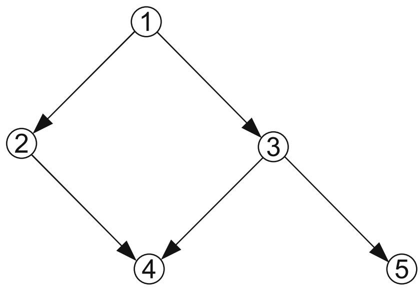

Fig. 6.1 Hierarchy used in Example 6.16

From the graph it is clear that authorizing sets for individual players are just collections of players on paths in the graph from that particular player to the principal executive in the hierarchy, player 1. We can deduce that

$$\mathfrak{A}_H(\{4\}) = \{\ \{1, 2, 4\},\ \{1, 3, 4\}\ \} \subset \Psi_H.$$

This immediately shows that the intersection of two disjunctively autonomous coalitions does not have to be autonomous. Namely, $\{1, 4\} = \{1, 2, 4\} \cap \{1, 3, 4\} \notin \Psi_H$.

To illustrate autonomous parts of a particular coalition we take the coalition $S = \{1, 2, 5\} \notin \Psi_H$. It is evident that its disjunctively autonomous part is given by $\delta_H(S) = \{1,2\} \in \Psi_H$. This illustrates the general property that for every non-autonomous coalition $S \notin \Psi_H$ we have that $\delta_H(S) \subsetneq S$. ■

Let $v \in \mathcal{G}^N$ be a cooperative game on the set of players N. As with the conjunctive approach, we assume that $v(S)$ represents the potential output of the coalition $S \subset N$ in case this coalition is able to form. However, if S is not disjunctively autonomous in the permission structure H, the authority exercised prevents the coalition S from forming. In this respect, S is not an institutional coalition in the sense that it does not have the institutional make-up to organize itself. In fact only its autonomous part $\delta_H(S) \subset S$ is able to form within H.

As for the conjunctive approach, I now introduce a mapping $\mathfrak{P}_H : \mathcal{G}^N \to \mathcal{G}^N$, which assigns to every cooperative game $v \in \mathcal{G}^N$ its *disjunctive restriction*, given as the game $\mathfrak{P}_H(v) \in \mathcal{G}^N$ with

$$\mathfrak{P}_H(v)(S) := v(\delta_H(S)), \quad S \subset N. \tag{6.16}$$

Obviously, for the empty permission structure H_0 on N it holds that $\mathfrak{P}_{H_0}(v) = v$ for all $v \in \mathcal{G}^N$.

The proof of the next characterization of the disjunctive restriction is omitted, since it is very similar to the proof of Theorem 6.9.

Theorem 6.17 *Let $H \in \mathfrak{H}^N_w$ be some strict permission structure. The disjunctive restriction $\mathfrak{P}_H$ for H is a linear projection mapping of rank $\#\Psi_H - 1$ on $\mathcal{G}^N$, where $\#\Psi_H - 1$ is the number of nonempty disjunctively autonomous coalitions in H. Its kernel (or null space) is spanned by the standard basis games $\{b_S \mid S \notin \Psi_H\}$ and its image is spanned by the unanimity basis games $\{u_S \mid S \in \Psi_H\}$.*

Next I investigate the dividends of the coalitions within the disjunctive restriction of a certain game in a given strict permission structure. Authorizing covers turn out to be of crucial importance in this analysis.

Let $S \subset N$ be a coalition. I define $\mathfrak{A}^*_H(S) \subset 2^N$ as the collection of all finite unions of authorizing covers of the coalition S, i.e., $T \in \mathfrak{A}^*_H(S)$ if and only if there exist $T_q \in \mathfrak{A}_H(S)$ $(1 \leqq q \leqq Q)$ such that $T = \bigcup_{q=1}^{Q} T_q$. It is clear that $\mathfrak{A}_H(S) \subset \mathfrak{A}^*_H(S) \subset \Psi_H$.

The disjunctive analogue of Theorem 6.10 can now be formulated as follows:

Theorem 6.18 *Let $v \in \mathcal{G}^N$ be some cooperative game. Then its disjunctive restriction on $H \in \mathfrak{H}^N_w$ is given by*

$$\mathfrak{P}_H(v) = \sum_{S \in \Psi_H} \left\{ \sum_{T \in \mathfrak{A}^{-1}_H(S)} \Delta_v(T) + \sum_{T \in \widehat{\mathfrak{A}}_H(S)} \mu^H_S(T) \cdot \Delta_v(T) \right\} \cdot u_S, \tag{6.17}$$

where

(i) $\mathfrak{A}^{-1}_H(S) := \{T \subset N \mid S \in \mathfrak{A}_H(T)\}$,

(ii) $\widehat{\mathfrak{A}}_H(S) := \{T \subset N \mid S \in \mathfrak{A}^*_H(T) \setminus \mathfrak{A}_H(T)\}$, *and*
(iii) *for every* $S \in \Psi_H$ *and* $T \in \mathfrak{A}^*_H(S) : \mu^H_S(T) = \Delta_{w_T}(S) \in \mathbb{Z}$ *with* $w_T = \mathfrak{P}_H(u_T)$ *and* $\mathbb{Z}$ *the collection of all whole numbers.*

A proof of Theorem 6.18 is included in the appendix of this chapter.

The numbers $\mu^H_S(T) \in \mathbb{Z}$ for coalitions $S, T \subset N$ as introduced in Theorem 6.18 are clearly independent of the game v, and therefore determined completely within the permission structure H. I state here as an unproven conjecture that ion general $\mu^H_S(T) \in \{-1, 0, 1\}$ for all $S, T \in 2^N$. In certain cases this can be confirmed and a formula for these numbers can indeed be derived. To illustrate this, an analysis for certain number $\mu^H_S(T)$ is provided in Section 6.5, the second appendix to this chapter. However, a general proof of the conjecture formulated here has not yet been developed and remains an open problem.

6.1.2.1 A Computational Approach

I complete the discussion of the disjunctive restriction of a cooperative game by focussing on the actual computation or determination of the disjunctive restriction of a given game. We use the multilinear extension (MLE) introduced in Chapter 1 of these lecture notes to establish the desired formulations.

I recall that the MLE of a cooperative game $v \in \mathcal{G}^N$ is given by a function $E_v : [0,1]^N \to \mathbb{R}$ with[5]

$$E_v(x) = \sum_{S \subset N} \Delta_v(S) \cdot \left\{ \prod_{I \in S} x_i \right\}. \tag{6.18}$$

Recall that E_v is a multilinear function, which coincides with the worth $v(S)$ at the extreme points of the unit cube $[0,1]^N \subset \mathbb{R}^N_+$. As discussed in Chapter 1, one can interpret the MLE of a game as a probabilistic expectation. Namely, we can rewrite $E_v(x_1, \ldots, x_n) = \mathbb{E}[v(\beth)]$, where $\beth$ is a random variable whose values are subsets of N given the probabilities $\text{Prob}\{i \in \beth\} = x_i$ and under the assumptions that the n events $\{i \in \beth\}$ are stochastically independent.

Next, I derive the MLE of the disjunctive restriction $\mathfrak{P}_S(u_S)$ of the unanimity game u_S for $S \subset N$. With this in mind we introduce two operators. The operator $\otimes$ is denoted as *independent multiplication* and is completely characterized by the following properties:

(i) For every $(x_1, \ldots, x_n) \in [0,1]^N$ and all $S, T \subset N$:

[5] The formulation used below is actually derived from the original definition of the MLE of a game. For details I refer to Equation (1.25) and its discussion in Chapter 1.

$$\left(\prod_{i \in S} x_i\right) \otimes \left(\prod_{i \in T} x_i\right) = \prod_{i \in S \cup T} x_i. \tag{6.19}$$

(ii) Let $f, g, h: [0,1]^N \to \mathbb{R}$ be three multilinear functions on $[0,1]^N$. Then

$$(f+g) \otimes h = (f \otimes h) + (g \otimes h) \quad \text{and} \tag{6.20}$$

$$f \otimes (g+h) = (f \otimes g) + (f \otimes h). \tag{6.21}$$

The second operator is denoted as *disjunctive addition* $\oplus$ and for every two multilinear functions $f, g: [0,1]^N \to \mathbb{R}$ it is defined by

$$f \oplus g = 1 - (1-f) \otimes (1-g). \tag{6.22}$$

Let $\beth$ be the random coalition variable with independent probabilities $(x_1, \ldots, x_n) \in [0,1]^N$. Now for all $S, T \subset N$ define the events $\mathbf{A} = \{S \subset \beth\}$ and $\mathbf{B} = \{T \subset \beth\}$. Then $\text{Prob}\{\mathbf{A}\} = \prod_{i \in S} x_i$ and $\text{Prob}\{\mathbf{B}\} = \prod_{j \in T} x_j$. We conclude from the above that[6]

$$\text{Prob}\,(\mathbf{A} \wedge \mathbf{B}) = \text{Prob}\{(S \cup T) \subset \beth\} = \prod_{i \in S \cup T} x_i = \text{Prob}(\mathbf{A}) \otimes \text{Prob}(\mathbf{B}).$$

Similarly we derive that

$$\text{Prob}(\mathbf{A} \vee \mathbf{B}) = \text{Prob}(\mathbf{A}) \oplus \text{Prob}(\mathbf{B}).$$

Next let $H \in \mathfrak{H}_w^N$ be a strict permission structure. Now take $i \in N$ and take probabilities $x \in [0,1]^N$. We define $\mathcal{P}_i$ as the probabilistic event that there exists an authorizing cover for $\{i\}$ in the random coalition $\beth$ and, moreover, we introduce $\mathbf{P}_i(x) = \text{Prob}\,(\mathcal{P}_i)$ as the probability that $\mathcal{P}_i$ occurs given the individual participation probabilities x. Thus, $\mathbf{P}_i(x)$ is the probability there exists an authorizing cover for $\{i\}$ in the random coalition $\beth$ given the individual participation probabilities x.

Since such an authorizing cover is in fact a permission path from some $j_1 \in B_H$ to i, we know that $\beth$ contains such an authorizing cover for player i if and only if

(i) $i \in \beth$ and
(ii) there is at least one superior $j \in H^{-1}(i)$ which has an authorizing cover in $\beth$.

This leads to the conclusion that

$$\mathbf{P}_i(x) = \text{Prob}\left\{ (i \in \beth) \wedge \left(\vee_{j \in H^{-1}(i)} \mathcal{P}_j \right) \right\}, \tag{6.23}$$

or

[6] The operator $\wedge$ stands for the logical "and", while the operator $\vee$ represents the logical "and/or".

$$\mathbf{P}_i(x) = x_i \otimes \left(\bigoplus_{j \in H^{-1}(i)} \mathbf{P}_j(x) \right). \tag{6.24}$$

Since for $j \in B_H$ it simply holds that $\mathbf{P}_j(x) = x_j$ we now have derived a recursive method for computing the multilinear function $\mathbf{P}_i(x)$, $i \in N$, which expresses the probability that i has an authorizing cover in $\beth$.

Finally from the computational rules it follows that for any coalition $S \subset N$ the probability that S has an authorizing set in $\beth$ is given by

$$\mathbf{P}_S(x) = \bigotimes_{i \in S} \mathbf{P}_i(x). \tag{6.25}$$

We thus conclude that the MLE of $\mathfrak{P}_H\,(u_S)$, where u_S is the unanimity game corresponding to S, is exactly given by the multilinear function $\mathbf{P}_S$. Since we now have the MLE of the disjunctive restriction of any unanimity game on H, we therefore have the MLE of the disjunctive restriction of any game v to the strict permission structure H.

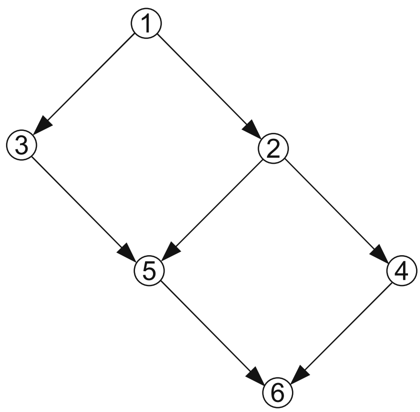

Fig. 6.2 Hierarchy used in Example 6.19

To illustrate this computational method, consider the permission structure given in Fig. 6.2.

Example 6.19 Take $N = \{1, \ldots, 6\}$ as the set of players and let $H \in \mathfrak{H}_w^N$ be the strictly hierarchical permission structure described in Fig. 6.2.

To compute $\mathfrak{P}_H\left(u_{\{6\}}\right)$ I follow the outlined recursive procedure for computing all multilinear functions $\mathbf{P}_i$, $i \in N$. I arrive at the following expressions:

$$\begin{aligned}
\mathbf{P}_1(x) &= x_1\\
\mathbf{P}_2(x) &= x_1x_2\\
\mathbf{P}_3(x) &= x_1x_3\\
\mathbf{P}_4(x) &= x_1x_2x_4\\
\mathbf{P}_5(x) &= x_5[\mathbf{P}_2(x) \oplus \mathbf{P}_3(x)] =\\
&= x_1x_3x_5 + x_1x_2x_5 - x_1x_2x_3x_5\\
\mathbf{P}_6(x) &= x_6[\mathbf{P}_4(x) \oplus \mathbf{P}_5(x)] =\\
&= x_1x_3x_5x_6 + x_1x_2x_5x_6 + x_1x_2x_4x_6 - x_1x_2x_3x_5x_6 - x_1x_2x_4x_5x_6
\end{aligned}$$

As argued before $\mathbf{P}_6$ is the MLE of $\mathfrak{P}_H\left(u_{\{6\}}\right)$ and thus

$$\mathfrak{P}_H\left(u_{\{6\}}\right) = u_{\{1,3,5,6\}} + u_{\{1,2,5,6\}} + u_{\{1,2,4,6\}} - u_{\{1,2,3,5,6\}} - u_{\{1,2,4,5,6\}}.$$

Now in $\mathfrak{P}_H\left(u_{\{6\}}\right)$ we have that $N \in \mathfrak{A}^*_H(\{6\})$. Also, $N \notin \mathfrak{A}_H(\{6\})$. From the formula given in Section 6.5 it now would follow that $\mu^H_N(\{6\}) \in \{-1, 1\}$. However, from the expression above and the fact that $\Delta_{u_{\{6\}}}(S) = 0$ if $S \neq \{6\}$. Thus, it has to be concluded that $\mu^H_N(\{6\}) = 0$. This shows that even in relatively simple cases, that do not satisfy the requirement as formulated in the appendix, we have a refutation of the conjecture that in general $\mu^H_T(S) \in \{-1, 1\}$ for arbitrary $S, T \subset N$. Note that this case is certainly not refuting the general conjecture that $\mu^H_T(S) \in \{-1, 0, 1\}$. ■

6.1.2.2 Some Applications

I consider two examples with some further applications of the disjunctive approach to games with a permission structure and the computational method developed above. The first example revisits the case of middle management already explored in Example 6.13.

Example 6.20 Again consider the specific hierarchical production organization with middle management developed in Example 6.13. As before we let $N = \{p, w\} \cup M$ where $M = \{1, \dots, m\}$ is the set of middle-tier managers. On N we define the game $v = u_{\{w\}}$ as the unanimity game for $\{w\}$. Also, H is given by $H(p) = M$, $H(j) = \{w\}$ for $j \in M$, and $H(w) = \varnothing$. Clearly $H \in \mathfrak{H}^N_s$.

We can now compute the disjunctive restriction of v using the method developed above. The MLEs for the different players are given by

$$\begin{aligned}
\mathbf{P}_p(x) &= x_p\\
\mathbf{P}_i(x) &= x_px_i \quad \text{where } i \in M\\
\mathbf{P}_w(x) &= \sum_{k=1}^{m}(-1)^{k+1}\sum_{S\subset M:|S|=k} x_px_w\prod_{i\in S}x_i
\end{aligned}$$

Obviously, $\mathbf{P}_w$ is the MLE of $\mathfrak{P}_H(v)$. And, so,

$$v'' = \mathfrak{P}_H(v) = \sum_{k=1}^{m} (-1)^{k+1} \sum_{S \subset M : |S| = k} \widehat{u}_S, \tag{6.26}$$

where for every coalition of middle managers $S \subset M$ we define

$$\widehat{u}_S(T) = \begin{cases} 1 \text{ if} \left[S \cup \{p, w\} \right] \subset T \\ 0 \text{ otherwise.} \end{cases}$$

The consequences of competition between the middle managers in M can be shown by computation of the Shapley value $\varphi_j(v'')$ for $j \in N$. I derive for the executive player p and the unique productive worker w that

$$\varphi_p\left(v''\right) \equiv \varphi_w\left(v''\right) = \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k+2} \frac{m!}{k!\,(m-k)!} = \frac{m(m+3)}{2(m+1)(m+2)},$$

and for every middle manager $i \in M$

$$\varphi_i\left(v''\right) = \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k+2} \frac{(m-1)!}{(k-1)!\,(m-k-1)!} = \frac{2}{m(m+1)(m+2)}.$$

This shows that for larger m the Shapley values of players p and w are increasing, while the Shapley values of the competing middle managers $i \in M$ are diminishing. The diminishing returns to the set of middle-tier managers is denoted as the contestability of the middleman position of these managers. ■

This example refers to a property that is unique to the disjunctive approach to hierarchical organization structure, namely *middleman contestability*. This refers to the property that if there are multiple superiors controlling a single subordinate, this subordinate is able to extract higher values from using the permission of one superior "against" the others. In other words, the superiors have a contested position in the hierarchy; the subordinate only requires the permission of a single superior to generate his value. As such each superior has a diminished role, reflected in the multiplicity of disjunctive authorizing sets for that subordinate. This reduces the dividends of these authorizing sets, allocating more value and power to that subordinate.

It is clear that the disjunctive approach differs only from the conjunctive approach if there are multiple superiors over certain groups of players in the hierarchy under consideration. In many applications such hierarchies are avoided. For example, in many theories of hierarchical production organizations ("firms") the hierarchies are either an extremely straightforward line-hierarchy[7] (Rajan and Zingales, 1998) or

[7] A *line hierarchy* consists of a single authority path, from the executive player to a single player in the lowest tier of the hierarchy.

a tree such that each player has exactly one superior (Radner, 1993; Van Zandt, 1999). In both cases there is no distinction between the conjunctive and disjunctive approaches to describing the exercise of authority in games with a permission value.

The next example discusses a very simple and straightforward application of the disjunctive approach to the analysis of a market organization versus a hierarchical production organization in the sense of Coase (1937). It is shown that the disjunctive approach essentially provides a set of good reasons why hierarchical authority organizations are strongly Pareto superior to market organizations with transaction costs when it comes to production—exactly as argued by Coase (1937) in his perspective on the firm.

Example 6.21 In this example I again consider a very simple production situation. As before, we assume a given production technology which converts one (composite) input into one output. All incomes are generated by the sale of this single output on an unspecified commodity market. Throughout I assume that the production technology is proprietary. In fact, again I assume that the production technology is owned by a single given player, who as a consequence has complete control over its use.

I assume that the owner of the production technology can achieve a certain level of output by herself given by $\varrho > 0$. Additional output is achieved by obtaining additional units of the input. This situation can be represented by the specifically designed TU-game $w \in \mathcal{G}^N$ in which the player set is given by $N = \{0\} \cup P$, where $0 \in N$ is the owner of the production technology as a user of the given production process and $i \in P$ are owners of additional units of input. Now a cooperative game-theoretic description of this production situation is given by w with

$$w = \varrho \cdot u_0 + \sum_{i \in P} u_{0i}, \tag{6.27}$$

where u_0, respectively u_{0i}, is the unanimity game with respect to $\{0\}$, respectively $\{0, i\}$. Obviously, the input of player $i \in P$ generates one additional unit of the output.

I modify the game w in different fashions to describe two organization structures, one through a costly market mechanism and one through a disjunctive hierarchy.

Case A: A Market Organization

First, I consider a market organization with one-sided transaction costs. The owner of the production facility 0 is assumed to purchase additional units of the input market with one-sided transaction costs. Here the owners of the input $i \in P$ are subjected to a cost of $c > 0$ for each unit sold. On the other hand, the owner 0 of the production process has no explicit market transaction costs. The output of the production organization in this situation can be described by the game

$$v_1 := w - c \sum_{i \in P} u_i, \tag{6.28}$$

where u_i is the unanimity game for $\{i\}$ with $i \in P$. I assume that for all input providers $i \in P$ the transaction costs are the same.

The expected payoff in the situation that all transactions take place through the market mechanism is now given by the Shapley value of the game v_1, i.e.,

$$\varphi_0(v_1) = \varrho + \frac{p}{2}, \text{ and } \varphi_i(v_1) = \frac{1}{2} - c, \quad i \in P,$$

where we let $p = |P| \geqq 1$. Here I explicitly state that the Shapley value φ is used as a risk-neutral utility function on the probabilistic outcomes of the costly market mechanism described. I refer for this Rothian interpretation of the Shapley value to the discussion in Chapter 3.

Case B: A Disjunctive Hierarchy

An alternative organization form would be to separate ownership and control of the production technology—as a representation of the productive asset itself— and allow the input market participants $i \in P$ to enter into an organization structure involving their partial control of the production technology. This allows the input providers to share in the revenues generated through the sale of the owner's output ϱ.

Such a hierarchical production organization is described by an acyclic permission structure $S{:}N' \to 2^{N'}$, where $N' = \{a, b\} \cup P$. Here, player a is the separated owner of the production technology, while player b stands for the controlled production technology itself. Thus, the original player 0—representing the owner using the production technology to generate ϱ units of output—is separated into an "owner" (a) and the production technology itself (b). All productivity of the owner using the production technology is now attributed to artificial player b. If every input provider $i \in P$ enters the hierarchical production organization I can now describe the resulting hierarchy H by[8]

$$H(i) = \begin{cases} P & \text{if } i = a \\ \{b\} & \text{if } i \in P \\ \emptyset & \text{if } i = b \end{cases}.$$

The permission structure S gives exactly the same structure as discussed in the previous example in this section. Thus, it is natural to expect that there will be competition between the second tier managers $i \in P$ over the leadership of the production technology b.

From the description it is clear that owners of the input take partial control of the production process itself. Now the output of this hierarchical production organization is described by $v_2 := \mathfrak{P}_H(w')$, where

[8] An alternative approach would be to allow the separated owner to directly control the production technology. Thus, $H'(a) = P \cup \{b\}$ in stead of the given $H(a) = P$. In H' the owner would reap even higher benefits from the implementation of the disjunctive hierarchy, although the analysis would technically lead to exactly the same qualitative conclusions. Only the explicit bounds on the variables for which the disjunctive hierarchy would be Pareto dominant would be different.

$$w' = \varrho \cdot u_b + \sum_{i \in P} u_{ib} \tag{6.29}$$

is a modification of the original game w given in Equation (6.27) by replacing 0 with b. This replacement indicates the actual separation of ownership and control with regard to the production process. Essentially I assume that the separated owner is not productive, but has the top position in the hierarchy. Now

$$\varphi_a(v_2) = \varphi_b(v_2) = \varrho \cdot \frac{p(p+3)}{2(p+1)(p+2)} + \frac{p}{3}, \quad \text{and} \tag{6.30}$$

$$\varphi_i(v_2) = \frac{2\varrho}{p(p+1)(p+2)} + \frac{1}{3}, \quad i \in P. \tag{6.31}$$

It is clear that the owner of the production technology shares the revenues of her own production activities with the providers of the inputs in this hierarchy. This is at the foundation of the following analysis.

A Comparison Between These Organizations

With the use of the Shapley values as evaluations of the different production organizations we now are able to give a comparison. In particular one may identify when the disjunctively hierarchical production situation with separation of ownership and control is Pareto superior to the market mechanism with one-sided transaction costs.

1. The owner of the production process has to gain by allowing the disjunctive organization of the production. This is the case when

   $$\varphi_0(v_1) < \varphi_a(v_2) + \varphi_b(v_2).$$

   This is equivalent to

   $$\varrho < \frac{1}{12} p(p+1)(p+2). \tag{6.32}$$

2. An individual input supplier $i \in P$ has to gain from entering the hierarchical production organization. This is the case when

   $$\varphi_i(v_1) < \varphi_i(v_2).$$

   This results into the requirement that

   $$\varrho > \left(\tfrac{1}{12} - \tfrac{1}{2}c \right) p(p+1)(p+2). \tag{6.33}$$

From the analysis above we deduce that for any transaction cost $c > 0$ and any output level $\varrho > 0$ the organization of production by a disjunctive hierarchy is weakly Pareto superior to a direct organization of production through the market mechanism if condition (6.32) is satisfied. This the case if the number of middle-tier managers p is large enough. Furthermore, a disjunctive hierarchy is even strongly Pareto superior to a market organization if

$$\left(\tfrac{1}{12}-\tfrac{1}{2}c\right)p(p+1)(p+2) < \varrho < \tfrac{1}{12}p(p+1)(p+2). \tag{6.34}$$

Again the main reason for this insight is middleman contestability, which is exploited in this example. ■

6.2 Shapley Permission Values

Consider a cooperative game with a permission structure (N, v, H). In the previous section I considered two fundamentally different approaches to the incorporation of the exercise of authority in such a situation. This resulted into the conjunctive restriction $\mathfrak{R}_H(v)$ and the disjunctive restriction $\mathfrak{P}_H(v)$ of the cooperative game v describing the potential values that coalitions can generate.

In this section I consider the Shapley values of these two restrictions. Together with René van den Brink, I introduced these specific Shapley values as *hierarchical permission* values. The Shapley value of the conjunctive restriction, $\zeta_H(v) = \varphi(\mathfrak{R}_H(v))$, is known as the conjunctive permission value, while the Shapley value of the disjunctive restriction, $\xi_H(v) = \varphi(\mathfrak{P}_H(v))$, is known as the disjunctive permission value.

In this section I debate several axiomatizations of these permission values. These axiomatizations are especially designed around properties that describe the nature of the exercise of authority in hierarchical organizations.

6.2.1 The Conjunctive Permission Value

The conjunctive permission value on the space of games with a permission structure is defined as the Shapley value of its conjunctive restriction. Hence, for the game v with permission structure H the conjunctive permission structure is defined by

$$\rho^c(v, H) = \varphi(\mathfrak{R}_H(v)). \tag{6.35}$$

The conjunctive permission value has been introduced seminally in an application in Gilles et al. (1992) and subsequently axiomatized in van den Brink and Gilles (1996); in that paper a seminal axiomatization of the conjunctive permission value was developed. Subsequent papers have developed alternative axiomatizations. Here I initially follow the discussion presented in van den Brink and Gilles (1996) and subsequently I will turn to the discussion of some alternative axiomatizations of ρ^c.

First, I reformulate the conjunctive permission value.

Proposition 6.22 *Let* $(N, v, H) \in \mathcal{G}^N \times \mathcal{H}^N$ *be some game with a permission structure. Then its conjunctive permission value is given by*

$$\rho_i^c(v, H) = \sum_{S \in \Gamma_H(i)} \frac{\Delta_v(S)}{\#\gamma_H^-(S)} \tag{6.36}$$

where $\Gamma_H(i) = \{S \mid S \cap H^+(i) \neq \varnothing\}$ *for every* $i \in N$ *and* $\gamma_H^-(S)$ *is the conjunctive authorizing cover of the coalition S.*

Proof Recall from Theorem 6.10 that

$$\mathfrak{R}_H(v) = \sum_{S \in \Gamma_H} \left(\sum_{T \subset N : \gamma_H^-(T) = S} \Delta_v(T) \right) \cdot u_S .$$

From the additivity property of the Shapley value it then follows that

$$\varphi_i(\mathfrak{R}_H(v)) = \sum_{S \in \Gamma_H : i \in S} \; \sum_{T \subset N : \gamma_H^-(T) = S} \frac{\Delta_v(T)}{\#S} .$$

Since (i) $S \in \Gamma_H$ if and only if $S = \gamma_H^-(S)$ if and only if there exists $T \subset N$ with $\gamma_H^-(T) = S$, and (ii) $i \in \gamma_H^-(T)$ if and only if $H^+(i) \cap T \neq \varnothing$ if and only if $T \in \Gamma_H(i)$, we arrive at the asserted relationship. ■

I examine an arbitrary allocation rule $f : \mathcal{G}^N \times \mathcal{H}^N \to \mathbb{R}^N$ on the class of games with a permission structure. I consider five axioms on such the allocation rule f to characterize the conjunctive permission value.

Theorem 6.23 *An allocation rule* $f : \mathcal{G}^N \times \mathcal{H}^N \to \mathbb{R}^N$ *is equal to the conjunctive permission value* ρ^c *if and only if f satisfies the following five axioms:*

Efficiency *For every* $v \in \mathcal{G}^N$ *and every* $H \in \mathcal{H}^N$ *it holds that*

$$\sum_{i \in N} f_i(v, H) = v(N). \tag{6.37}$$

Additivity *For all* $v, w \in \mathcal{G}^N$ *and* $H \in \mathcal{H}^N$ *it holds that*

$$f(v + w, H) = f(v, H) + f(w, H). \tag{6.38}$$

Weakly inessential player property *For every* $v \in \mathcal{G}^N$, *every* $H \in \mathcal{H}^N$ *and every player* $i \in N$ *such that every player* $j \in H^+(i)$ *is a dummy player*[9] *in* v *it holds that* $f_i(v, H) = 0$.

Necessary player property *For every monotone game* $v \in \mathcal{G}^N$, $H \in \mathcal{H}^N$ *and player* $i \in N$ *with* $v(S) = 0$ *for all* $S \subset N \setminus \{i\}$ *it holds that*

$$f_i(v, H) = \max_{j \in N} f_j(v, H). \tag{6.39}$$

Structural monotonicity *For every monotone game* $v \in \mathcal{G}^N$, $H \in \mathcal{H}^N$ *and player* $i \in N$ *such that* $H(i) \neq \varnothing$ *it holds that*

$$f_i(v, H) \geqq \max_{j \in H(i)} f_j(v, H). \tag{6.40}$$

A proof of Theorem 6.23 is provided in the appendix to this chapter.

Some discussion of the axioms developed in van den Brink and Gilles (1996) and presented in Theorem 6.23 is warranted. The first two axioms of efficiency and additivity are variations of the standard axioms used by Shapley (1953) in his seminal axiomatization of his value.

The *weakly inessential player property* requires that a player is assigned a minimal payoff if she and all of her subordinates are dummy players in the given game. Note that a player is no longer "inessential" if one of her subordinates contributes to the generation of economic values. In that case a player has managerial control of that productive player and, therefore, is herself part of a productive chain of command.

The *necessary player property* reverses the logic of the weakly inessential player property. Indeed, a player is "necessary" if she is essential in the generation of any of the cooperative economic values; i.e., without this necessary player there are no values generated. Such necessary players attain the maximal value in the productive hierarchy as described by the game with the permission structure under consideration.

Finally, *structural monotonicity* requires that every player is assigned a payoff that is at least as large as any of her direct subordinates. This property is akin to the Hierarchical Payoff Property discussed in Chapter 5 for directed networks. It is easy to see that structural monotonicity can be reformulated as that for every monotone game $v \in \mathcal{G}^N$, every $H \in \mathcal{H}^N$ and player $i \in N$ such that $H(i) \neq \varnothing$ it holds that

$$f_i(v, H) = \max_{j \in H^+(i)} f_j(v, H). \tag{6.41}$$

Hence, every player is assigned a payoff as least as large as any of her (indirect) subordinates.

[9] Recall that a player i is a dummy player in v if $V(S) = v(S \setminus \{i\})$, i.e., the player i does not contribute any thing to the cooperative game v.

The independence of the five stated properties can be shown through a series of five examples:

(a) Let $f^1: \mathcal{G}^N \times \mathcal{H}^N \to \mathbb{R}^N$ be defined by

$$f_i^1(v,H) = 0 \tag{6.42}$$

Then the allocation rule f^1 satisfies all properties except efficiency.

(b) First, for every (v,H) define

$$X(v,H) = \left\{ i \in N \;\middle|\; \mathfrak{R}_H(v)(N-i) = \min_{j \in N} \mathfrak{R}_H(v)(N-j) \right\}$$

as the set of players with maximal marginal contributions to the grand coalition N in (v,H). Now, let $f^2: \mathcal{G}^N \times \mathcal{H}^N \to \mathbb{R}^N$ be defined by

$$f_i^2(v,H) = \begin{cases} \frac{v(N)}{\#X(v,H)} & \text{if } i \in X(v,H) \\ 0 & \text{otherwise} \end{cases} \tag{6.43}$$

Then the allocation rule f^2 satisfies all properties except additivity.

(c) Let (v,H) be such that H contains at least one cycle. Let $C(H) \subset N$ be the collection of all players in a cycle in H. Now define

$$v^H(S) = \begin{cases} \mathfrak{R}_H(v)(S) & \text{if } C(H) \subset S \\ 0 & \text{if } C(H) \setminus S \neq \varnothing \end{cases}$$

Now, let $f^3: \mathcal{G}^N \times \mathcal{H}^N \to \mathbb{R}^N$ be defined by

$$f^3(v,H) = \varphi\left(v^H\right). \tag{6.44}$$

Then the allocation rule f^3 satisfies all properties except the weakly essential player property.

(d) Let $f^4: \mathcal{G}^N \times \mathcal{H}^N \to \mathbb{R}^N$ be defined by

$$f_i^4(v,H) = \sum_{j \in H^+(i)} \frac{\rho_j^c(v,H)}{\#H^-(j)} \tag{6.45}$$

Then the allocation rule f^4 is a re-allocation of the conjunctive permission value and satisfies all properties except the necessary player property.

(e) Finally, let $f^5: \mathcal{G}^N \times \mathcal{H}^N \to \mathbb{R}^N$ be defined as the regular Shapley value of the game v, i.e.,

$$f^5(v,H) = \varphi(v) \tag{6.46}$$

Then the allocation rule f^5 satisfies all properties except structural monotonicity.

The axiomatization introduced in van den Brink and Gilles (1996) has a definite drawback in the sense that the properties used in this axiomatization cannot be applied straightforwardly to arrive at a similar axiomatization of the so-called disjunctive permission value—defined as the Shapley value of the disjunctive restriction of a game with an acyclic permission structure. In two contributions René van den Brink developed an alternative approach based on two variations of Myerson (1977)'s fairness property that allows a uniform axiomatization of both the conjunctive as well as the disjunctive permission values. (van den Brink, 1997, 1999) This is the subject of the following discussion.

I first discuss the axiomatization of the conjunctive permission value based on a *conjunctive fairness* property introduced in van den Brink (1999). The property of conjunctive fairness addresses the change of the allocated values if a hierarchical authority relationship between a superior and her subordinate is removed from the permission structure. The main hypothesis reflected in the fairness axiom is that the superior and subordinate are affected equally. However, conjunctive fairness imposes that this equal treatment extends to certain third parties.

The main additional hypothesis introduced in this discussion is the restriction of the conjunctive permission value ρ^c to the class of strictly hierarchical permission structures $\mathfrak{H}_s^N$. Without this deliberate restriction the introduced concept of fairness cannot be implemented properly.

First, I introduce some auxiliary notation. Throughout let $(v, H) \in \mathcal{G}^N \times \mathfrak{H}_s^N$ is a game with a strictly hierarchical permission structure. For $i \in N$ I now denote $\overline{H}(i) \subset H^+(i)$ as the set of players that are *completely controls* by player i in the hierarchy H in the sense that $j \in \overline{H}(i)$ if and only if $j \in H^+(i)$ and every path from i_H to j includes the player i.[10]

Clearly, the set $\overline{H}^{-1}(i) = \{j \in N \mid i \in \overline{H}(j)\}$ now denotes the collection of players that completely control player i. From the definitions it should be clear that this collection is of particular importance with regard to the authorization of the activities involving player i.

I am now in the position to introduce two fundamental properties that can used in various axiomatizations of Shapley permission values:

Weak structural monotonicity For every monotone game with a strictly hierarchical permission structure $(v, H) \in \mathcal{G}^N \times \mathfrak{H}_s^N$ and player $i \in N$ it holds that $f_i(v, H) \geqq f_j(v, H)$ for all $j \in \overline{H}(i)$.

Conjunctive fairness For every game with a strictly hierarchical permission structure $(v, H) \in \mathcal{G}^N \times \mathfrak{H}_s^N$ and players $i, j, g \in N$ such that $i \neq g$ and $j \in H(i) \cap H(g)$ it holds that for every $h \in \{g\} \cup \overline{H}^{-1}(g)$:

[10] Formally, player i completely controls player j in H if for every path $i_1, \ldots, i_K$ with $i_1 = i_H$, $i_K = j$, and $i_{k+1} \in H(i_k)$ it holds that $i \in \{i_1, \ldots, i_K\}$. Note that by definition $\overline{H}(i_H) = N$, i.e., the executive i_H completely controls every other player in the hierarchy H.

$$f_h(v,H) - f_h(v,H_{-ij}) = f_j(v,H) - f_j(v,H_{-ij}) \tag{6.47}$$

where H_{-ij} is the permission structure resulting from eliminating the authority relation between players i and j.

Some interpretation and discussion of these three properties seems appropriate.

Weak structural monotonicity imposes that a player receives a payoff that is at least as large as any of the players that she completely dominates. Note that structural monotonicity required that a player receives a payoff at least as large as any of her subordinates, not just the players that she completely dominates. Clearly, structural monotonicity implies weak structural monotonicity.

Conjunctive fairness imposes that if a player has multiple direct superiors and the authority relationship with one of these superiors is removed, the remaining superiors as well as all of the players that completely dominate these remaining superiors receive the same change in their payoffs as that this subordinate. This conjunctive fairness property addresses how the allocated payoffs to remaining superiors are affected if some authority relationship is removed.

Founded on these ideas, and conjunctive fairness in particular, van den Brink (1999) constructed the following axiomatization of the conjunctive permission value. For a proof of this axiomatization I again refer to the appendix of this chapter.

Theorem 6.24 *An allocation rule* $f\colon \mathcal{G}^N \times \mathfrak{H}_s^N \to \mathbb{R}^N$ *on the class of games with strictly hierarchical permission structures is equal to the conjunctive permission value* ρ^c *if and only if* f *satisfies efficiency, additivity, the weakly inessential player property, the necessary player property, weak structural monotonicity, as well as conjunctive fairness.*

The strength of this particular axiomatization of the conjunctive permission value is the link it provides with the disjunctive permission value. In particular how the conjunctive permission value differs from its disjunctive counter-part. This is the subject of the next subsection.

On the other hand further axiomatizations are possible. The conjunctive permission value is an example of an allocation rule on a class of rather complex allocation problems—the class of cooperative games with a permission structure—that allows a very complex axiomatization. The two axiomatizations discussed here are based on five, respectively six, independent axioms. This is rather significant and allows for further development and deepening of these axiomatizations. I will not pursue this subject any further here; instead, I refer the interested readers to van den Brink (2006). In that paper further development of these axiomatizations is pursued and more complex sets of properties are discussed.

6.2.2 The Disjunctive Permission Value

The disjunctive permission value on the space of games with a strictly hierarchical permission structure $\mathcal{G}^N \times \mathfrak{H}_s^N$ is defined as the Shapley value of its disjunctive restriction. Hence, for the game v with permission structure H the *disjunctive permission value* is defined by

$$\rho^d(v,H) = \varphi(\mathfrak{P}_H(v)). \tag{6.48}$$

The disjunctive permission value has seminally been introduced in an example in Gilles and Owen (1999) and subsequently axiomatized in van den Brink (1997). As pointed out in the previous discussion, this axiomatization is closely related to the axiomatization of the conjunctive permission value developed in van den Brink (1999).

As usual we can provide a formula of the disjunctive permission value using the unanimity decomposition of the disjunctive restriction of a game provided in Theorem 6.18. Unfortunately, this decomposition contains multipliers ($\mu_S^H(T) = \Delta_{\mathfrak{P}_H(u_T)}(S)$) of an unknown size that are hard to determine. Therefore, this decomposition is not very useful for this purpose. A better computational method for the disjunctive permission value is to apply the computational method to determine the multi-linear extension (MLE) of the game first and then using this MLE to compute the disjunctive permission value of the game with the permission structure under consideration.

The axiomatization of the disjunctive permission value avoids these technicalities and can be founded on the same properties as used in the characterization of the conjunctive permission value. I need to introduce a modification of conjunctive fairness to establish such a characterization:

> *Disjunctive fairness* For every game with a strictly hierarchical permission structure $(v,H) \in \mathcal{G}^N \times \mathfrak{H}_s^N$ and players $i,j \in N$ such that $j \in H(i)$ and $\#H^{-1}(j) \geqq 2$ it holds that for every $h \in \{i\} \cup \overline{H}^{-1}(i)$:
>
> $$f_h(v,H) - f_h(v,H_{-ij}) = f_j(v,H) - f_j(v,H_{-ij}) \tag{6.49}$$
>
> where H_{-ij} is the permission structure resulting from eliminating the authority relation between players i and j.

Disjunctive fairness is a much simpler property than conjunctive fairness; it considers how the payoffs of the involved superior is affected if an authority relationship with one of her direct subordinates is removed. Hence, disjunctive fairness requires that the allocated payoffs to a superior and the players that completely dominate her are affected in exactly the same fashion as the payoff to the subordinate with whom the authority relationship is removed.

On the other hand we are able to provide an axiomatization of the disjunctive permission value that is based on the approach developed in van den Brink (1997) and van den Brink (1999) based on the idea of "fairness" in the allocation of value in a hierarchical authority structure.

Theorem 6.25 *An allocation rule* $f : \mathcal{G}^N \times \mathfrak{H}_s^N \to \mathbb{R}^N$ *on the class of games with strictly hierarchical permission structures is equal to the disjunctive permission value* ρ^d *if and only if f satisfies efficiency, additivity, the weakly inessential player property, the necessary player property, weak structural monotonicity, as well as disjunctive fairness.*

For a proof of this axiomatization I refer to the appendix of this chapter.

To illustrate the properties of the disjunctive permission structure consider the following example:

Example 6.26 Consider $N = \{1,2,3,4\}$ and the strictly hierarchical permission structure H depicted in Fig. 6.3. thus, the permission structure depicted is given as $H(1) = \{2,3\}$, $H(2) = H(3) = \{4\}$, and $H(4) = \varnothing$.

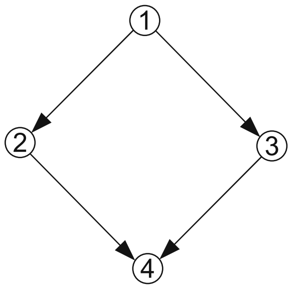

Fig. 6.3 Hierarchy H used in Example 6.26

Next consider the simple unanimity game $v = u_{\{4\}}$. Then the conjunctive restriction of (v, H) is given by $\mathfrak{R}(v, H) = u_N$, the unanimity game of the grand coalition N. On the other hand, the disjunctive restriction of (v, H) is given by

$$\mathfrak{P}(v,H)(S) = \begin{cases} 1 & \text{if } S = \{1,2,4\},\ S = \{1,3,4\} \text{ or } S = N \\ 0 & \text{otherwise.} \end{cases}$$

From this it is clear that

$$\rho^c(v,H) = \left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right) \tag{6.50}$$

$$\rho^d(v,H) = \left(\tfrac{5}{12}, \tfrac{1}{12}, \tfrac{1}{12}, \tfrac{5}{12}\right) \tag{6.51}$$

Next remove the authority relation of players 3 and 4 in the hierarchy H. We thus arrive at the modified hierarchical permission structure H' depicted in Fig. 6.4.

For the modified hierarchy H' it is easy to determine that

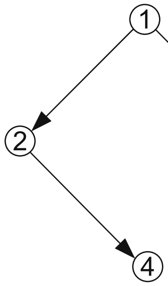

Fig. 6.4 Modified hierarchy H' from Example 6.26

$$\mathfrak{R}(v,H') = \mathfrak{P}(v,H') = u_{\{1,2,4\}}$$

and that, consequently,

$$\rho^c(v,H') = \rho^d(v,H') = \left(\tfrac{1}{3}, \tfrac{1}{3}, 0, \tfrac{1}{3}\right) \tag{6.52}$$

We can now consider the fairness of the treatment of players 3 and 4 with regard to the removal of the authority relation $(3,4)$. I compute

$$\begin{aligned} \rho^c_3(v,H) - \rho^c_3(v,H') = \tfrac{1}{4} - 0 = \tfrac{1}{4} \neq -\tfrac{1}{12} = \tfrac{1}{4} - \tfrac{1}{3} = \rho^c_4(v,H) - \rho^c_4(v,H') \\ \rho^d_3(v,H) - \rho^d_3(v,H') = \tfrac{1}{12} - 0 = \tfrac{1}{12} = \tfrac{5}{12} - \tfrac{1}{3} = \rho^d_4(v,H) - \rho^d_4(v,H') \end{aligned}$$

Hence, the disjunctive permission value has a built-in fairness with regard to the treatment of players 3 and 4 concerning the removal of the authority relation between them. On the other hand, the conjunctive permission value treats both players in exactly the opposite fashion. ■

To complete the discussion of the axiomatization of the disjunctive permission value given in Theorem 6.25 we have to determine examples of allocation rules that show that the used axioms are independent.

(a) Consider the conjunctive permission value $\rho^c : \mathcal{G}^N \times \mathfrak{H}^N_s \to \mathbb{R}^N$ on the class of games with strictly hierarchical permission structures. Then from our previous discussion it is clear that ρ^c satisfies all axioms except the disjunctive fairness axiom.
(b) Let $f^1 : \mathcal{G}^N \times \mathfrak{H}^N_s \to \mathbb{R}^N$ be given by $f^1(v,H) = \varphi(v)$. Then f^1 satisfies all axioms except weak structural monotonicity.
(c) Next let $f^2 : \mathcal{G}^N \times \mathfrak{H}^N_s \to \mathbb{R}^N$ be given by

$$f^2_i(v,H) = \begin{cases} v(N) & \text{if } i = i_H \\ 0 & \text{if } i \neq i_H. \end{cases}$$

This allocation rule satisfies all axioms of Theorem 6.25 except for the necessary player property.

(d) Define $f^3 : \mathcal{G}^N \times \mathfrak{H}_s^N \to \mathbb{R}^N$ by $f^3(v, H) = \frac{1}{n} v(N)$. Then f^3 satisfies all axioms except the weakly inessential player property.

(e) Define $w_T \in \mathcal{G}^N$ for $T \subset N$ given by

$$w_T(S) = \begin{cases} 1 & \text{if } S \cap T \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

Next let $f^4 : \mathcal{G}^N \times \mathfrak{H}_s^N \to \mathbb{R}^N$ be given by

$$f^4(v, H) = \begin{cases} f^2(v, H) & \text{if } v = w_T \text{for some } T \subset N, \#T \geqq 2 \\ \rho^d(v, H) & \text{otherwise.} \end{cases}$$

This allocation rule satisfies all desired axioms except additivity.

(f) Finally, let $f^5 : \mathcal{G}^N \times \mathfrak{H}_s^N \to \mathbb{R}^N$ be given by $f_i^5(v, H) = 0$ for all $i \in N$. Then f^5 satisfies all axioms used in Theorem 6.25 except efficiency.

6.3 Modeling Economic Phenomena

At the conclusion of this chapter and also of this book, I consider the implications of the use of cooperative game theoretic instruments to describe certain categories of economic phenomena.

First, it can be argued that standard market transactions, trade relations, and other forms of direct communication and economic interaction can be represented through binary relationships. Therefore, undirected networks form the tools of choice for the representation of such socio-economic activities as trade, markets with indivisible commodities (housing), disease transmission, and other social interaction such as marriage and social discourse. These activities can be understood to have primarily a *non-cooperative* nature. These phenomena, therefore, are naturally relegated to the realm of non-cooperative game. This non-cooperative approach has indeed been well-developed in an extensive literature to date; I refer to Goyal (2007) and Jackson (2008) for an overview of network formation theories using non-cooperative game-theoretic tools. I also refer to Vega-Redondo (2007) and Jackson (2008, Chapter 5) for an account of more advanced statistical and physics-based theories of complex social networks.

However, even in the setting of undirected networks it is natural to consider the cooperative activities of players within the network under consideration. Here, I refer in particular to the class of equilibrium notions based on the concept of *pairwise stability*, seminally introduced and developed by Jackson and Wolinsky (1996). Pairwise stability incorporates how pairs of players contemplate the formation of a new link between them as well as how individual players decide about the breaking of the links under their control.[11]

Pairwise stability can be categorized as a hybrid concept that brings together certain non-cooperative as well as cooperative game-theoretic features within the setting of undirected network formation. It combines the standard individual rationality arguments of Nash equilibrium with the ability to block the formation of a network by pairs of players through their consideration of creating new links. Thus, a restricted, Core-like equilibrium concept emerges in which individual players as well as pairs of players are allowed to form as institutional coalitions. On the other hand, the settings of a network formation situation are fundamentally different from a standard game in characteristic function form; in particular, the payoff structure is more general, since payoffs are not assigned to coalitions of players, but rather to social networks of players.[12] This implies that the comparison is imperfect at best. For further details and discussion I also refer to Jackson (2008, Chapter 6).

In the discussion above I limited my remarks to the study of undirected networks in which links are formed by equals. However, relationships between unequal parties, in particular authority relations, form a major part of social functionality in our society as well. Through a series of examples in this chapter, I have pointed out that authority relations are paramount in production situations that are founded on the organization of a division of labor. Only through the exercise of authority can such specialization and a division of labor be made to work properly. This was mathematically shown in Chapter 5 where directed communication with the expectation of a higher return for the communicator—or predecessor—results into a fully hierarchical organization of this directed communication. Hierarchical authority organizations are therefore the most natural organizational forms for production activities based on a division of labor requiring directed communication between the various specialists. In short, authority relations are particularly useful to control and direct cooperative activities in which instructions are passed from one party to another.

The idea that cooperative activities are best described using authority relations and hierarchical organizations emanated from the seminal work by Coase (1937). Coase set out to explain why hierarchical production organizations—simply known as *firms*—exist in the first place. He argued that the cost of using the market or price mechanism itself—founded on the writing and execution of complete, market-based labor contracts—supports the emergence of a hierarchical production organization. Such transaction costs indeed cause those non-cooperative relations to be replaced by cooperative relations based on an incomplete contract in which the exercise of

[11] Since the inception of the notion of pairwise stability, extensions have been proposed and links with more traditional concepts such as Pareto efficiency have been explored. Certain extensions of the pairwise stability concepts can be related to the modeling of trust in network formation (Gilles and Sarangi, 2009).

[12] Thus, network formation theory encompasses how players are interacting. As such the network structure directly enters the payoff functions as an argument and variable. This is a major departure and generalization of the original framework considered by Myerson (1977), who considered restrictions of cooperative games in characteristic function form.

authority replaces the determining details of the required tasks in the complete market contract.

More recently Hart and Moore (1990) amended the Coaseian hypothesis by only considering the most fundamental form of authority to be exercised. Thus, they only consider the authority of a superior to exclude a subordinate from accessing a productive asset. (This was explored fully in this chapter through Examples 6.13, 6.20, and 6.21.) This reduces the exercise of authority to the enforcement of a property right related to this productive asset. This corresponds exactly to the fashion in which authority is modelled in the conjunctive as well as the disjunctive approach: superiors can veto the subordinate to access a productive asset. The two approaches only differ on how this veto right is distributed if multiple superiors are involved.

Alchian and Demsetz (1972) introduced the seminal idea of "team production", the expression of the ultimate cooperative activity in the economy and they made explicit that output based on incomplete contracts is best represented using cooperative games. Again this fundamental descriptive tool is widely used in the literature on the nature of the firm. Also in the current chapter, I model output levels based on team production described through some cooperative game.

The only aspect that is less well represented in the approach pursued in the current chapter is that of the *agency relationship*. The idea of the agency relationship is firmly grounded in the foundation of production on specialization and the implementation of a division of labor. Indeed, the superior in an agency relationship is viewed as less knowledgeable than the subordinate, who is a fully qualified specialist who has to perform a productive task. This introduces the so-called *agency problem*: The superior—or principal—has insufficient knowledge of the task at hand to fully supervise and evaluate the subordinate—or agent—who is charged to execute this task. The information deficiency of the principal has two sides to it. First, she has insufficient information to select a maximally performing agent (the *adverse selection* problem) and, second, she has not sufficient information to evaluate the quality of the performed work to fully assess the agent's work (the *moral hazard* problem). For a seminal contribution in the theory of the agency problem, I refer to Williamson (1975). From the seminal work by Coase, Alchian and Demsetz, and Williamson there has emerged a substantial field of research in economics that tries to address the explanation of the nature of the firm. Cooperative game theory has a central role in this field.

It is clear that the theory developed in this chapter does not do justice to the agency problem. In fact, the authority relationships considered are too simple to represent agency relationships. The theory therefore has to be expanded to incorporate the division of labor that is present within the hierarchical organization and model the agency problem into the model. This is a formidable task.

I conclude the discussion of the theory developed in this chapter with some pointers for expanding the scope and extend of the developed theory to incorporate such issues as the agency problem.

In van den Brink and Gilles (2009) we address the possibility to expand the essentially cooperative theory of team production in an authority hierarchy to a behavioral model of productive activities. At the foundation of this theory lies the conjunctive approach which is used to describe the generated team output levels if authority is fully exercised. Based on these values an assessment can be given whether superiors within the hierarchy should exercise that authority or not. van den Brink and Gilles (2009) consider two possible ways to exercise authority, explicitly and latently.

When authority is exercised explicitly, superiors make the deliberate decision to enforce the employment contract with a subordinate and to supervise this subordinate's access to the productive asset at a certain cost. This naturally results into a standard non-cooperative game in which the strategies of the superiors are simply the subsets of subordinates that she wishes to supervise and the outcomes are the conjunctively restricted team production levels that result given the exercised supervision. In the resulting Nash equilibria in this non-cooperative game, superiors will exercise complete control—through comprehensive supervision—at a minimal overall cost.

Authority is exercised latently, if subordinates voluntarily submit to the conditions of the employment contract. Thus, under latent authority none of the superiors have to exercise oversight explicitly at a certain cost. It is clear that latent authority allows the hierarchical production organization to operate under zero organizational costs.

Latent authority is established if the subordinates in the hierarchical organization are sufficiently rational to anticipate the supervising activities of their superiors. This can be done through the application of more advanced equilibrium concepts from non-cooperative game theory in the model discussed above. Indeed, if subordinates are endowed with a system of beliefs, then these beliefs can reflect when exactly their superiors will actively supervise their productive activities. Rational subordinates therefore will anticipate such supervision to occur and will decide to perform their tasks as if they were actively supervised by their superiors. The result of this behavior is that the hierarchy will be self-enforced, i.e., full latent authority is exercised.

From the discussion above it should be clear that the theory of authority developed in the current chapter can indeed be used in a fruitful way to expand into a complete game-theoretic model of the firm.

6.4 Appendix: Proofs of the Main Theorems

Proof of Proposition 6.7

From the definition of the conjunctive autonomous part it is immediately clear that for every coalition $S \subset N$ it holds that

$$\gamma(S) = \{i \in S \mid H^{-}(i) \subset S\}. \tag{6.53}$$

Using this equality we can now show the desired properties:

(i) Let $i \in \gamma(S) \cap \gamma(T)$. Then either $H^-(i) \subset S$ or $H^-(i) \subset T$ or both. Thus, $H^-(i) \subset (S \cup T)$, implying the assertion.
(ii) Clearly, $i \in \gamma(S \cap T)$ if and only if $H^-(i) \subset S \cap T$. This is equivalent to the statement that $i \in \gamma(S)$ as well as $i \in \gamma(T)$.
(iii) This assertion follows immediately from the following equation:

$$\begin{aligned}\gamma^-(S \cup T) &= \bigcup_{i \in S \cup T} H^-(i) \cup S \cup T \\ &= \left[\bigcup_{i \in S} H^-(i)\right] \cup \left[\bigcup_{i \in T} H^-(i)\right] \cup S \cup T \\ &= \gamma^-(S) \cup \gamma^-(T).\end{aligned}$$

(iv) For $i \in \gamma^-(S \cap T)$ it either holds that $i \in S \cap T$ or there is some $j \in S \cap T$ with $j \in H^+(i)$. If $i \in S \cap T$, then surely $i \in \gamma^-(S)$ as well as $i \in \gamma^-(T)$, i.e., $i \in \gamma^-(S) \cap \gamma^-(T)$.
If there is some $j \in S \cap T$ with $j \in H^+(i)$, then by the fact that $j \in S$ as well as $j \in T$ it is concluded that $i \in \gamma^-(S)$ as well as $i \in \gamma^-(T)$. This implies again that $i \in \gamma^-(S) \cap \gamma^-(T)$.

Proof of Theorems 6.9 and 6.10

The proof of the assertions stated in Theorems 6.9 and 6.10 is done in several steps. These intermediate results have some interest in their own right.

Lemma 6.27 *Let $S \subset N$ with $S \neq \varnothing$ be any non-empty coalition. Then*

$$\mathfrak{R}_H(u_S) = u_{\gamma^-(S)}. \tag{6.54}$$

Proof Let $T = \gamma^-(S)$ and $w = \mathfrak{R}_H(u_S)$. Evidently T is an autonomous coalition, i.e., $\gamma(T) = T$. Denote by $V \subset N$ be any coalition.

First consider the case that $T \subset V$. Then $S \subset T = \gamma(T) \subset \gamma(V)$, and therefore $w(V) = u_S(\gamma(V)) = 1$.

Next suppose that $T \setminus V \neq \varnothing$. Then there exists some player $j \in T$ with $j \notin V$. Since $j \in T$ we have that either $j \in S$ or $j \in H^-(S)$. If $j \in S$, then $S \setminus V \neq \varnothing$ and therefore $E \setminus \gamma(V) \neq \varnothing$. If $j \in H^-(S)$, then there exists some player $i \in S$ with $i \in H^+(j)$. As $j \notin V$, this implies that $i \in H^+(N \setminus V)$ and so $i \notin \gamma(V)$. Again we arrive at the conclusion that $S \setminus \gamma(V) \neq \varnothing$.

From the conclusion that $S \setminus \gamma(V) \neq \varnothing$, it is derived that $w(V) = u_S(\gamma(V)) = 0$.

The two main conclusions now lead to the statement that

$$w(V) = \begin{cases} 1 & \text{if } T = \gamma^-(S) \subset V \\ 0 & \text{otherwise} \end{cases}$$

Hence, we conclude that $w = u_{\gamma^{-}(S)}$. ■

It is clear that the assertion stated in Theorem 6.10 now follows immediately from Lemma 6.27. For a proof of Theorem 6.9 we have the following arguments:

Suppose that the coalition S is not autonomous. Let $w = \mathfrak{R}_H(b_S)$ be the conjunctive restriction of the standard basis game for S.

Now there is no coalition T such that $S = \gamma(T)$. Thus for any coalition $T \subset N$ (including $T = S$) we conclude that $w(T) = b_S(\gamma(T)) = 0$. Hence, w is the null game and is in the kernel of $\mathfrak{R}_H$.

Now suppose that $S \in \Gamma_H$ with $S \neq \varnothing$ is a non-empty autonomous coalition. By Lemma 6.27 it follows that $\mathfrak{R}_H(u_S) = u_{\gamma^{-}(S)}$ and hence that u_S is indeed in the image of the mapping $\mathfrak{R}_H$.

Let $A = \#\Gamma_H - 1$. Since there are $\dim\left(\mathcal{G}^N\right) - (\#\Gamma_H - 1) = 2^n - 1 - A$ non-autonomous coalitions, there are at least $2^n - 1 - A$ independent games in the kernel of $\mathfrak{R}_H$. So, the dimension of this kernel has to be at least $2^n - 1 - A$.

Similarly, there are A non-empty autonomous coalitions. Thus, there are A independent games in the image of $\mathfrak{R}_H$. Therefore, the dimension of this image has to be at least A.

On the other hand, the sum of these dimensions has to be precisely $2^n - 1$. This implies that these computed dimensions are exact. This also results into the conclusion that the bases for the kernel and the image are indeed correct.

To see that $\mathfrak{R}_H$ is a projection mapping, we note that if v is in the image of $\mathfrak{R}_H$, then v can be expressed as

$$v = \sum_{S \in \Gamma_H} c_S \cdot u_S.$$

Hence, from Lemma 6.27 it immediately can be concluded that $\mathfrak{R}_H(v) = v$. This shows that $\mathfrak{R}_H$ is indeed a projection on its image.

Proof of Theorem 6.11

I follow the proof of the assertion developed in van den Brink and Gilles (2009).

First, we show that the restriction mapping $\mathfrak{R}$ indeed satisfies the five properties stated in the assertion. Let $H \in H^N$ and $v, w \in \mathcal{G}^N$. Since $\gamma_H(N) = N$ it holds that $\mathfrak{R}(v, H)(N) = v(\gamma_H(N)) = v(N)$, and thus $\mathfrak{R}$ satisfies property (i).

$\mathfrak{R}$ satisfies (ii) since $\mathfrak{R}(v + w, H)(S) = (v + w)(\gamma_H(S)) = v(\gamma_H(S)) + w(\gamma_H(S)) = \mathfrak{R}(v, H)(S) + \mathfrak{R}(w, H)(S)$ for all $S \subset N$.

If $i \in N$ is such that all $j \in H^{+}(i) \cup \{i\}$ are null players in v then $\mathfrak{R}(v, H)(S) = v(\gamma_H(S)) = v(\gamma_H(S) \setminus (\{i\} \cup H^{+}(i))) = v(\gamma_H(S \setminus \{i\})) = \mathfrak{R}(v, H)(S \setminus \{i\})$ for all $S \subset N$, and thus $\mathfrak{R}$ satisfies property (iii).

If $i \in N$ is such that $v(S) = 0$ for all $S \subset N \setminus \{i\}$, then for $S \subset N \setminus \{i\}$ we have that $i \notin \gamma_H(S)$ and thus $\mathfrak{R}(v, H)(S) = v(\gamma_H(S)) = 0$, which in turn implies that $\mathfrak{R}$ satisfies property (iv).

Finally, property (v) follows from the fact that $j \in H(i)$ and $S \subset N \setminus \{i\}$ implies that $\gamma_H(S) = \gamma_H(E \setminus \{j\})$ and thus $\mathfrak{R}(v, H)(S) = v(\gamma_H(S)) = v(\gamma_H(S \setminus \{j\})) = \mathfrak{R}(v, H)(S \setminus \{j\})$.

Next suppose that $\mathfrak{F}: \mathcal{G}^N \times \mathcal{H}^N \to \mathcal{G}^N$ satisfies the five listed properties, and let $H \in \mathcal{H}^N$. Consider the game $w_T = c_T u_T$ with $c_T \geqq 0$ and u_T the unanimity game of $T \subset N$.

Property (i) now implies that $\mathfrak{F}(w_T, H)(N) = c_T$. Define $\alpha_H(T) = T \cup H^-(T)$. We distinguish the following cases with respect to $S \subset N$, $S \neq N$:

- $T \not\subset S$. Since for all players $i \in T$ it holds that $w_T(S) = 0$ for all $S \subset N \setminus \{i\}$, property (iv) implies that $\mathfrak{F}(w_T, H)(S) = 0$.
- $T \subset S$, $\alpha_H(T) \not\subset S$. Then there exists a sequence of players $(h_1, \ldots h_p)$ such that $h_1 \in \alpha_H(T) \setminus S$, $h_k \in H(h_{k-1})$ for all $k \in \{2, \ldots, p\}$, and $h_p \in T$. Property (iv) and repeated application of property (v) now imply that $\mathfrak{F}(w_T, H)(E) = \mathfrak{F}(w_T, H)(S \setminus \{j\}) = 0$.
- $\alpha_H(T) \subset S$. Since for all agents $i \in N \setminus \alpha_H(S)$ it holds that all $j \in H^+(i) \cup \{i\}$ are null players in w_T, property (iii) implies that $\mathfrak{F}(w_T, H)(S) = \mathfrak{F}(w_T, H)(N) = c_T$.

So, we may conclude that $\mathfrak{F}(w_T, H) = \mathfrak{R}(w_T, H)$. The assertion stated in Theorem 6.11 then follows with property (ii) and the fact that the game v can be expressed as a linear combination of the unanimity games u_T in a unique fashion.

Proof of Theorem 6.18

The proof of Theorem 6.18 is conducted through a series of lemmas. In the sequel we take a fixed coalition $S \subset N$. Furthermore, we define $w_S := \mathfrak{P}_H(u_S)$ as the disjunctive restriction of the unanimity game u_S on the acyclic permission structure $H \in \mathfrak{H}_w^N$ on N.

Lemma 6.28 *For every non autonomous coalition $T \notin \Psi_S$ with the property that $S \subset \delta_H(T): \Delta_{w_S}(T) = 0$.*

Proof By definition it holds that

$$\Delta_{w_S}(T) = \sum_{R \subset T} (-1)^{|T|-|R|} w_S(R).$$

Let $V := T \setminus \delta_H(T)$. Clearly $V \cap S = \varnothing$. Since T is not autonomous it is evident that $V \neq \varnothing$.

Next take $j \in V$ and let $R \subset T \setminus \{j\}$.

If $S \subset \delta_H(R)$, then clearly $w_S(R) = w_S(R \cup \{j\}) = 1$.
If $S \setminus \delta_H(R) \neq \varnothing$, then obviously $w_S(R) = w_S(R \cup \{j\}) = 0$.

This shows that for every $R \subset T \setminus \{j\}$:

$$w_S(R) - w_S(R \cup \{j\}) = 0.$$

Now rewrite

$$\begin{aligned}\Delta_{w_S}(T) &= \sum_{R \subset T} (-1)^{|T|-|R|} w_S(R) \\ &= \sum_{R \subset T \setminus \{j\}} \left((-1)^{|T|-|R|} w_S(R) + (-1)^{|T|-|R|-1} w_S(R \cup \{j\}) \right) \\ &= \sum_{R \subset T \setminus \{j\}} (-1)^{|T|-|R|} \left(w_S(R) - w_S(R \cup \{j\}) \right) = 0.\end{aligned}$$

This shows the assertion. ■

Lemma 6.29 *For every coalition* $T \in \mathfrak{A}_H(S) : \Delta_{w_S}(T) = 1$.

Proof For every strict subset R of T it holds that either $R \notin \Psi_H$ or $S \setminus R \neq \varnothing$. (This is a consequence of the definition of the collection $\mathfrak{A}_H(S)$.)

In both cases it follows that $S \setminus \delta_H(R) \neq \varnothing$. This implies that for every strict subcoalition R of T it holds that $w_S(R) = 0$ and so $\Delta_{w_S}(R) = 0$. Furthermore, $w_S(T) = u_S(\delta_H(T)) = u_S(T) = 1$, which implies that

$$\Delta_{w_S}(T) = w_S(T) - \sum_{R \subset T : R \neq T} \Delta_{w_S}(R) = 1.$$

This shows the assertion. ■

Lemma 6.30 *For every coalition* $T \notin \mathfrak{A}^*_H(S) : \Delta_{w_S}(T) = 0$.

Proof Define $\widehat{T} := \cup\{R \in \mathfrak{A}_H(S) \mid R \subset T\}$. By definition it holds that $\widehat{T} \subset F$.

For the case that $\widehat{T} = \varnothing$ it is evident that $w_S(T) = u_S(\delta_H(T)) = 0$ since then from the definition of autonomous covers $S \setminus \delta_H(T) \neq \varnothing$. This immediately shows that $\Delta_{w_S}(T) = 0$.

So we may restrict ourselves to the case that $\widehat{T} \neq \varnothing$. Clearly $\widehat{T} \in \mathfrak{A}^*_H(S) \subset \Psi_H$. Hence, by the fact that $T \notin \mathfrak{A}^*_H(S)$ it follows that $T \setminus \widehat{T} \neq \varnothing$. But it also holds that $S \subset \widehat{T} \subset \delta_H(T)$.

If $T \notin \Psi_H$, then by Lemma 6.28 it holds that $\Delta_{w_S}(T) = 0$. So, we suppose that $T \in \Psi_S$, i.e., $T = \delta_H(T)$. Take $R := T \setminus \widehat{T} \neq \varnothing$ and let $j \in R$. Take $V \subset T \setminus \{j\}$. Then it is obvious that $S \subset \widehat{T} \subset \delta_H(V)$. So, $w_S(V) = u_S(\delta_H(V)) = 1$. It is also clear that $w_S(V \cup \{j\}) = 1$. We therefore may conclude that for every coalition $V \subset T \setminus \{j\}$ it holds that $w_S(V) - w_S(V \cup \{j\}) = 0$. Hence,

$$\begin{aligned}\Delta_{w_S}(T) &= \sum_{V \subset T} (-1)^{|T|-|V|} w_S(V) \\ &= \sum_{V \subset T \setminus \{j\}} (-1)^{|T|-|V|} \left(w_S(V) - w_S(V \cup \{j\}) \right) = 0.\end{aligned}$$

This shows the assertion. ■

With the use of the lemmas as stated and proved above we are able to prove Theorem 6.18.

Proof of Theorem 6.18 Let $v \in \mathcal{G}^N$ and define $w = \mathfrak{P}_H(v)$ as its disjunctive restriction on the strict permission structure $H \in \mathfrak{H}^N_w$. Using the unanimity basis, we can write the game v as

$$v = \sum_{S \subset N} \Delta_v(S) \cdot u_S.$$

Take any coalition $S \subset N$ then by combination of Lemma 6.28, Lemma 6.29, and Lemma 6.30 we derive that for $w_S := \mathfrak{P}_H(u_S)$ it holds that

$$w_S = \sum_{V \in \mathfrak{A}_H(S)} u_V + \sum_{V \in \mathfrak{A}^\star_H(S) : H \notin \mathfrak{A}_H(S)} \Delta_{w_S}(V) \cdot u_V.$$

Next define for every $V \in \mathfrak{A}^*_H(S)$

$$\mu^H_V(S) := \Delta_{w_S}(V) = \sum_{R \subset V} (-1)^{|V|-|R|} \cdot w_S(R),$$

then it is evident that these numbers are independent of the original game v. Moreover, since for every $R \subset N$ it holds that $w_S(R) \in \{0, 1\}$ we may conclude that these numbers are whole. This implies by linearity of the mapping $\mathfrak{P}_H$ that

$$\begin{aligned} w &= \sum_{S \subset N} \Delta_v(S) \cdot \mathfrak{P}_H(u_S) \\ &= \sum_{S \subset N} \Delta_v(S) \cdot \left\{ \sum_{V \in \mathfrak{A}_H(S)} u_V + \sum_{V \in \mathfrak{A}^\star_H(S) : V \notin \mathfrak{A}_H(S)} \mu^H_V(S) \cdot u_V \right\}. \end{aligned}$$

Rewriting this formula leads to

$$w = \sum_{S \in \Psi_H} \left\{ \sum_{T \in \mathfrak{A}^{-1}_H(S)} \Delta_v(T) + \sum_{T \in \widehat{\mathfrak{A}}_H(S)} \mu^H_S(T) \cdot \Delta_v(T) \right\} u_S.$$

This completes the proof of Theorem 6.18.

Proof of Theorem 6.23

First I show that ρ^c satisfies the five listed properties. *Efficiency* follows immediately from the definition and the fact that the Shapley value is efficient. *Additivity* similarly follows from additivity of the Shapley value and the fact that $\mathfrak{R}(v,H)+\mathfrak{R}(w,H)=\mathfrak{R}(v+w,H)$. Also, for every $h \in \mathcal{H}^N$ and games $v,w \in \mathcal{G}^N$ it holds that for every $S \subset N$:

$$\begin{aligned}\gamma_H(S)\setminus H^+(i) &= (S\setminus H^+(N\setminus S))\setminus H^+(i) = \\ &= (S\setminus\{i\})\setminus\left[H^+(N\setminus S)\cup H^+(i)\right] = \\ &= (S\setminus\{i\})\setminus H^+(N\setminus(S\setminus\{i\}) = \gamma_H(S\setminus\{i\}).\end{aligned} \tag{6.55}$$

To show the *weakly inessential player property*, let $i \in N$ be weakly inessential in (v,H). Since $j \in H^+(i)$ is a dummy in v, by (6.55), for every $S \subset N$

$$\begin{aligned}\mathfrak{R}_H(v)(S) &= v(\gamma_H(S)) = v(\gamma_H(S)\setminus H^+(i)) = \\ &= v(\gamma_H(S\setminus\{i\})) = \mathfrak{R}_H(v)(S\setminus\{i\})\end{aligned} \tag{6.56}$$

Thus, $\rho_i^c(v,H)=\varphi_i(\mathfrak{R}_H(v))=0$, showing the weakly inessential player property.

Let v be a monotone game. Assume that $i \in N$ is a necessary player in (v,H). Since $v(S)=0$ for all $S\subset N$ with $i\notin S$ and $\gamma_H(S)\subset S$, it holds that $0 \leqq \mathfrak{R}_H(v)(S) = v(\gamma_H(S)) \leqq v(S)=0$ for all $S\subset N\setminus\{i\}$. Therefore,

$$\rho_i^c(v,H)=\varphi_i(\mathfrak{R}_H(v)) \geqq \varphi_j(\mathfrak{R}_H(v)) = \rho_j^c(v,H)$$

for all $j \in N$. This implies the *necessary player property*.

To show *structural monotonicity*, let v be monotone. Let $i \in N$ and $j \in H(i)$. Then $\mathfrak{R}_H(v)(S\cup\{j\}) = \mathfrak{R}_H(v)(S)$ for all coalitions $S \subset N-ij$. Since v is monotone, $\mathfrak{R}_H(v)$ is monotone as well. by definition of the Shapley value it then follows again that $\rho_i^c(v,H)=\varphi_i(\mathfrak{R}_H(v)) \geqq \varphi_j(\mathfrak{R}_H(v)) = \rho_j^c(v,H)$. Thus ρ^c indeed satisfies structural monotonicity.

Next suppose that $f\colon\mathcal{G}^N\times\mathcal{H}^N\to\mathbb{R}^N$ satisfies the five stated properties.

Consider the unanimity game $w_S = c\cdot u_S$ for $S\subset N$ and $c \geqq 0$. Suppose that $S\in\Gamma_H$. Then with Proposition 6.22 it follows that

$$\rho_i^c(w_S,H)=\begin{cases}\frac{c}{\#\gamma_H^-(S)} & \text{if } i\in\gamma_H^-(S)\\ 0 & \text{otherwise}\end{cases}$$

I now show that $f(w_S,H)=\rho^c(w_S,H)$. For that I distinguish three cases with regard to a player $i \in N$:

Case 1: $i \in N \setminus \gamma_H^-(S)$ In this case both i and all $j \in H^+(i)$ are dummies in w_S. Thus, with the weakly inessential player property it follows that $f_I(w_S, H) = 0$.

Case 2: $i \in S$ This implies that i is necessary in w_S and by the necessary player property, $f_i(w_S, H) = \max_N f_j(w_S, H)$. In other words, there exists some constant $k \geqq 0$ with $f_i(w_S, H) = k$ for $i \in S$ and $f_j(w_S, H) \leqq k$ for $j \notin S$.

Case 3: $i \in \gamma_H^-(S) \setminus S$ Then $H(i) \neq \varnothing$ and, since w_S is monotone, with repeated application of structural monotonicity it follows that

$$f_i(w_S, H) \geqq \max_{j \in S(i)} f_j(w_S, H) = \max_{j \in H^+(i)} f_j(w_S, H).$$

From Case 2 and the fact that $H^+(i) \cap S \neq \varnothing$, we get $f_i(w_S, H) = k$.

Through Cases 1–3, I now have shown that

$$f_i(w_S, H) = \begin{cases} k & \text{if } i \in \gamma_H^-(S) \\ 0 & \text{otherwise} \end{cases}$$

Efficiency of f then implies that $k = \frac{c}{\#\gamma_H^-(S)}$.

This leads to the conclusion that $f = \rho^c$ on the class of nonnegative weighted unanimity games. Using additivity of f and ρ^c, we conclude that $f = \rho^c$ on the extended class of all weighted unanimity games and, therefore, all weighted sums of unanimity games. Hence, by the fact that the unanimity games form a basis of $\mathcal{G}^N$, we arrive at the conclusion that $f = \rho^c$ on $\mathcal{G}^N \times \mathcal{H}^N$.

Proof of Theorem 6.24

I first show that ρ^c satisfies the six listed axioms in the assertion. In the proof of Theorem 6.23 I already checked efficiency, additivity, the weakly inessential player property as well as the necessary player property. Since structural monotonicity implies weak structural monotonicity, I only have to show that ρ^c satisfies conjunctive fairness.

Again let $v_T = c \cdot u_T$ with $c \geqq 0$ and $T \subset N$ be a proportional unanimity game of coalition T. Let $w_T^H = \mathfrak{R}_H(v_T)$ be the conjunctive restriction of v_T on $H \in \mathfrak{H}_s^N$. Now for every coalition $S \subset N$ we introduce

$$d(S, H) = \frac{\Delta_{w_T^H}(S)}{\#S}$$

as the fairly divided Harsanyi dividend of the conjunctive restriction w_T.

Now let $h, j, g \in N$ with $h \neq g$ and $j \in H(h) \cap H(g)$. From the definition of the Shapley value φ and the fact that $\Delta_{w_T}(S) = 0$ for all $E \notin \Gamma_H$ it follows that

$$\rho^c_g(v_T, H_{-hj}) = \sum_{S \in \Gamma_{H_{-hj}} : g \in S} d(S, H_{-hj}) =$$
$$= \sum_{S \in \Gamma_H : g \in S} d(S, H_{-hj}) + \sum_{S \in \Gamma_{H_{-hj}} \setminus \Gamma_H : g \in S} d(S, H_{-hj})$$

Since $H^{-1}_{-hj}(i) = H^{-1}(i)$ if $i \neq j$, the following facts can be established:

1. If $S \in \Gamma_H$ and $\{j, g\} \not\subseteq S$ then the conjunctively autonomous part of $S' \subset S$ in H is the same as the conjunctively autonomous part of S' in H_{-hj}. Thus, $w^H_T(S') = \mathfrak{R}_{H_{-hj}}(v_T)(S') = w^{H_{-hj}}_T(S')$. Similarly, the Harsanyi dividends for the coalition S' under these restrictions is exactly the same.
2. In a similar fashion we can show that $S \in \Gamma_{H_{-hj}}$ and $\{g, j\} \not\subseteq S$ imply that $S \in \Gamma_H$.
3. Finally, $g \notin S$ and $j \in S$ imply that $S \notin \Gamma_{H_{-hj}}$.

Now with the above we can derive that

$$\begin{aligned}
\rho^c_g(v_T, H_{-hj}) - \rho^c_j(v_T, H_{-hj}) &= \sum_{S \in \Gamma_{H_{-hj}} : g \in S} d(S, H_{-hj}) - \sum_{S \in \Gamma_{H_{-hj}} : j \in S} d(S, H_{-hj}) \\
&= \sum_{S \in \Gamma_{H_{-hj}} : g \in S, j \notin S} d(S, H_{-hj}) - \sum_{S \in \Gamma_{H_{-hj}} : g \notin S, j \in S} d(S, H_{-hj}) \\
&= \sum_{S \in \Gamma_{H_{-hj}} \setminus \Gamma_H : g \in S, j \notin S} d(S, H_{-hj}) + \sum_{S \in \Gamma_H : g \in S, j \notin S} d(S, H_{-hj}) + \\
&\quad - \sum_{S \in \Gamma_{H_{-hj}} \setminus \Gamma_H : g \notin S, j \in S} d(S, H_{-hj}) - \sum_{S \in \Gamma_H : g \notin S, j \in S} d(S, H_{-hj}) \\
&= \sum_{S \in \Gamma_H : g \in S, j \notin S} d(S, H) - \sum_{S \in \Gamma_H : g \notin S, j \in S} d(S, H) \\
&= \sum_{S \in \Gamma_H : g \in S} d(S, H) - \sum_{S \in \Gamma_H : j \in S} d(S, H) \\
&= \rho^c_g(v_T, H) - \rho^c_j(v_T, H).
\end{aligned}$$

Also, by definition $S \in \Gamma_{H_{-hj}}$ and $g \in S$ imply that $\overline{H}^{-1}(g) \subset S$. Thus, as we derived for player g, we can now derive for any player $i \in \overline{H}^{-1}(g)$ that

$$\rho^c_i(v_T, H_{-hj}) - \rho^c_j(v_T, H_{-hj}) = \rho^c_i(v_T, H) - \rho^c_j(v_T, H)$$

Now for arbitrary games $v \in \mathcal{G}^N$ it now holds that

$$\mathfrak{R}(v, H)(S) = \sum_{T \subset N} \Delta_v(T) \cdot u_T(\gamma_H(S))$$

and, hence, the conjunctive permission value of (v, H) is the sum of Shapley values of the nature that we discussed above.

Next we have to prove that the six listed properties admit at most one allocation rule f, which then has to be the conjunctive permission value ρ^c. Hence, assume that the allocation rule $f: \mathcal{G}^N \times \mathfrak{H}_s^N \to \mathbb{R}^N$ satisfies the six listed properties.

Again consider $v_T = c \cdot u_T$ with $c \geqq 0$ and $T \subset N$. Let $w_T^H = \mathfrak{R}_H(v_T)$ be the conjunctive restriction of v_T on $H \in \mathfrak{H}_s^N$.

As in the proof of Theorem 6.23 the weakly inessential property now implies that $f_i(v_T, H) = 0$ if $i \in N \setminus \gamma_H^-(T)$, where $\gamma_H^-(T)$ is the authorizing cover of T in H.

Furthermore, the necessary player property and weak structural monotonicity imply that there exists some constant $k \geqq 0$ such that $f_i(v_T, H) = k$ for all $i \in \alpha(T)$, where

$$\alpha(T) = \left\{ j \in \gamma_H^-(T) \mid T \cap \left\{ \{i\} \cup \overline{H}(i) \right\} \neq \varnothing \right\}.$$

First, consider the minimal permission structure in the sense of a minimal number of authority relationships. In these cases the permission structure H represents a straight authority line with $\sum_{i \in N} |H(i)| = n - 1$ and $|H^{-1}(i)| = 1$ for all $i \neq i_H$. Thus, $\overline{H}(i) = H^+(i)$ for all $i \in N$. In these cases $T \cap \left\{ \{i\} \cup \overline{H}(i) \right\} \neq \varnothing$ for all $i \in \gamma_H^-(T)$ and, therefore, $\alpha(T) = \gamma_H^-(T)$. So, there exists some constant $k \geqq 0$ such that

$$f_i(v_T, H) = \begin{cases} k & \text{if } i \in \gamma_H^-(T) \\ 0 & \text{otherwise} \end{cases}$$

Efficiency now implies that $k = \frac{c}{\# \gamma_H^-(T)}$, and, therefore, $f(v_T, H) = \rho^c(v_T, H)$.

The proof now proceeds with induction on the number of authority relationships in the permission structure. Let $H \in \mathfrak{H}_s^N$ and assume by induction that $f(v_T, H') = \rho^c(v_T, H')$ for all $H' \in \mathfrak{H}_s^N$ such that $\sum_{i \in N} |H'(i)| < \sum_{i \in N} |H(i)|$.

Next define recursively the "level sets" $\{L_k \mid k = 0, 1, 2, \cdots\}$ of H by $L_0 = \varnothing$ and for all $k \in \mathbb{N}$:

$$L_p = \left\{ i \in N \setminus \cup_{t=0}^{p-1} L_t \;\middle|\; H(i) \subset \cup_{t=0}^{p-1} L_t \right\} \tag{6.57}$$

In van den Brink and Gilles (1994) it was shown that there exists some $M \in \mathbb{N}$ with $\{L_1, \ldots, L_M\}$ forming a finite partitioning of N consisting of non-empty sets only.

From the above, let $k \geqq 0$ be such that $f_i(v_T, H) = k$ for all $i \in \alpha(T)$. Our proof now continues with a procedure to determine the allocated values $f_i(v_T, H)$ as functions of k for all players $i \in N$.

For players $i \in L_1$, one of the two following conditions hold:

(a) If $i \in N \setminus \gamma_H^-(T)$, then $f_i(v_T, H) = 0$.

(b) If $i \in \gamma_H^-(T)$, then $i \in T$ since $H(i) = \varnothing$ and, thus, $f_i(v_T, H) = k$.

Next let $2 \leqq p \leqq M$ and suppose that we have determined the allocated values $f_j(v_T, H)$ for all players $j \in \cup_{t=1}^{p-1} L_t$ as a function of k. For every player $i \in L_p$ one of the following three conditions hold:

(a) If $i \in N \setminus \gamma_H^-(T)$, then $f_i(v_T, H) = 0$.
(b) If $i \in \alpha(T)$, then $f_i(v_T, H) = k$.
(c) If $i \in \gamma_H^-(T) \setminus \alpha(T)$, then by the definitions of $\gamma_H^-(T)$ and $\alpha(T)$ there exists some player $g \in \{i\} \cup \overline{H}(i)$ and some player $j \in H(g)$ such that $\# H^{-1}(j) \geqq 2$. Conjunctive fairness then implies that

$$f_i(v_T, H) - f_i(v_T, H_{-hj}) = f_j(v_T, H) - f_j(v_T, H_{-hj})$$

for $h \in H^{-1}(j) \setminus \{g\}$.
Since $\sum_{i \in N} |H_{-hj}(i)| = \sum_{i \in N} |H(i)| - 1$, with the induction hypothesis it then follows that

$$f_i(v_T, H) = f_j(v_T, H) + f_i(v_T, H_{-hj}) - f_j(v_T, H_{-hj}) \tag{6.58}$$

Since $j \in H^+(i)$ implies that $j \in L_q$ with $q < p$, this means that $f_j(v_T, H)$ is already expressed as a function of k. Similarly, by the induction hypothesis $f_i(v_T, H_{-hj}) = \rho_i^c(v_T, H_{-hj})$ and $f_j(v_T, H_{-hj}) = \rho_j^c(v_T, H_{-hj})$. This implies that $f_i(v_T, H)$ is now determined as a function of k.

Since $\{L_1, \ldots, L_M\}$ forms a partitioning of N consisting of non-empty sets only, the procedure described indeed determines all values $f_i(v_T, H)$ $(i \in N)$ as functions of k. Efficiency then uniquely determines the value k. Since the conjunctive permission value ρ^c satisfies the six listed axioms, it must hold that $f(v_T, H) = \rho^c(v_T, H)$.

Thus far we have shown that $f = \rho^c$ on the class of all nonnegative proportional unanimity games. To complete the proof, consider $v_T = c \cdot u_T$ with $c < 0$. Now, $v_T + (-v_T) = \eta$, where $\eta \in \mathcal{G}^N$ is the null game. Therefore, $\mathfrak{R}_H(v_T) + \mathfrak{R}_H(-v_T) = \mathfrak{R}_H(\eta) = \eta$. Therefore it follows that $f(v_T, H) = -f(-v_T, H) = -\rho^c(-v_T, H) = -\varphi(\mathfrak{R}_H(-v_T)) = -\varphi(-\mathfrak{R}_H(v_T)) = \varphi(\mathfrak{R}_H(v_T)) = \rho^c(v_T, H)$.

Finally, since every game $v \in \mathcal{G}^N$ can be expressed as a sum of proportional unanimity games, it then follows with additivity that $f(v, H) = \rho^c(v, H)$ for all (v, H). This completes the proof of the assertion.

Proof of Theorem 6.25

First I show that the disjunctive permission value ρ^d indeed satisfies the six stated axioms. For that purpose let $(v, H) \in \mathcal{G}^N \times \mathfrak{H}_s^N$.

Efficiency of ρ^d follows immediately from the efficiency of the Shapley value and the fact that $\delta_H(N) = N$.

Additivity follows from the additivity of the Shapley value and additivity of the disjunctive restriction operator $\mathfrak{P}_H$.

Let $i \in N$ be a weakly inessential player in (v, H). Then i is a null player in $\mathfrak{P}_H(v)$ and, thus, from the null player property of the Shapley value it then can be concluded that ρ^d indeed satisfies the *weakly inessential player property*.

We already established that the disjunctive restriction of a monotone game is monotone. Also the Shapley value can be written as

$$\varphi_j(v) = \sum_{S:j\in S} p(S) \cdot (\, v(S) - v(S-i)\,) \quad \text{where } p(S) = \frac{(n-|S|)!\,(|S|-1)!}{n!}.$$

Now suppose that v is monotone and $i \in N$ is a necessary player in v. Then i is also a necessary player in the monotone game $\mathfrak{P}_H(v)$. From this follows that

(i) $\mathfrak{P}_H(v)(S) - \mathfrak{P}_H(v)(S-i) = \mathfrak{P}_H(v)(S) \geqq \mathfrak{P}_H(v)(S) - \mathfrak{P}_H(v)(S-j)$ for all $j \in N$ and $S \subset N$;
(ii) $\mathfrak{P}_H(v)(S) - \mathfrak{P}_H(v)(S-i) \geqq 0$ for all S with $i \in S$, and
(iii) $\mathfrak{P}_H(v)(S) - \mathfrak{P}_H(v)(S-j) = 0$ for all $j \in N$ and S with $i \notin S$.

With the formula of the Shapley value it then follows that for every $j \neq i$

$$\begin{aligned}
\rho_i^d(v,H) = \varphi_i(\mathfrak{P}_H(v)) = \\
&= \sum_{S:i,j\in S} p(S) \cdot (\,\mathfrak{P}_H(v)(S) - \mathfrak{P}_H(v)(S-i)\,) + \\
&\qquad + \sum_{S:i\in S, j\notin S} p(S) \cdot (\,\mathfrak{P}_H(v)(S) - \mathfrak{P}_H(v)(S-i)\,) \geqq \\
&\geqq \sum_{S:i,j\in S} p(S) \cdot (\,\mathfrak{P}_H(v)(S) - \mathfrak{P}_H(v)(S-j)\,) + \\
&\qquad + \sum_{S:i\notin S, j\in S} p(S) \cdot (\,\mathfrak{P}_H(v)(S) - \mathfrak{P}_H(v)(S-j)\,) = \\
&= \varphi_j(\mathfrak{P}_H(v)) = \rho_j^d(v,H).
\end{aligned}$$

This shows that ρ^d indeed satisfies the *necessary player property*.

Next let v be monotone and let $i \in N$. From monotonicity of $\mathfrak{P}_H(v)$ it then follows that

(i) $\mathfrak{P}_H(v)(S) - \mathfrak{P}_H(v)(S-i) \geqq \mathfrak{P}_H(v)(S) - \mathfrak{P}_H(v)S-j)$ for all $j \in \overline{H}(i)$ and $S \subset N$ since $\delta_H(S-i) \subset \delta_H(S-j)$;
(ii) $\mathfrak{P}_H(v)(S) - \mathfrak{P}_H(v)(S-i) \geqq 0$ for all S with $i \in S$, and
(iii) $\mathfrak{P}_H(v)(S) - \mathfrak{P}_H(v)(S-j) = 0$ for all $j \in \overline{H}(i)$ and S with $i \notin S$.

Similarly as we constructed for showing the necessary player property, we can now apply the formula of the Shapley value to establish that ρ^d satisfies *weak structural monotonicity*.

Finally, we show that ρ^d satisfies *disjunctive fairness*. Let $w_T = C \cdot u_T$ for $T \subset N$ and some constant $C \in \mathbb{R}$. Let $j, h \in N$ such that h is one of at least direct superiors of j, i.e., $j \in H(h)$ and $H^{-1}(j) \geqq 2$. From the definition of the Shapley value and the fact that $\Delta_{\mathfrak{P}_{H(w_T)}}(S) = 0$ for all $S \notin \Psi_H$, it now follows that

$$\rho_i^d(w_T, H) = \sum_{S \in \Psi_H : i \in S} \frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|} = \\ = \sum_{S \in \Psi_{H_{-hj}} : i \in S} \frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|} + \sum_{S \in \Psi_H \setminus \Psi_{H_{-hj}} : i \in S} \frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|}$$

Now $S \in \Psi_H \setminus \Psi_{H_{-hj}}$ implies that $\{h, j\} \subset S$, which implies furthermore that

$$\sum_{S \in \Psi_H \setminus \Psi_{H_{-hj}} : h \in S} \frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|} = \sum_{S \in \Psi_H \setminus \Psi_{H_{-hj}} : j \in S} \frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|}.$$

Second, if $\{h, j\} \nsubseteq S$ then $\mathfrak{P}_H(w_T)(S') = \mathfrak{P}_{H_{-hj}}(w_T)(S')$ for all $S' \subset S$ and in particular $\Delta_{\mathfrak{P}_{H(w_T)}}(S') = \Delta_{\mathfrak{P}_{H_{-hj}}(w_T)}(S')$.

From this we can then derive that

$$\begin{aligned} \rho_h^d(w_T, H) - \rho_j^d(w_T, H) &= \sum_{S \in \Psi_{H_{-hj}} : h \in S} \frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|} - \sum_{S \in \Psi_{H_{-hj}} : j \in S} \frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|} \\ &= \sum_{S \in \Psi_{H_{-hj}} : h \in S, j \notin S} \frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|} - \sum_{S \in \Psi_{H_{-hj}} : j \in S, h \notin S} \frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|} \\ &= \sum_{S \in \Psi_{H_{-hj}} : h \in S, j \notin S} \frac{\Delta_{\mathfrak{P}_{H_{-hj}}(w_T)}(S)}{|S|} - \sum_{S \in \Psi_{H_{-hj}} : j \in S, h \notin S} \frac{\Delta_{\mathfrak{P}_{H_{hj}}(w_T)}(S)}{|S|} \\ &= \sum_{S \in \Psi_{H_{-hj}} : h \in S} \frac{\Delta_{\mathfrak{P}_{H_{-hj}}(w_T)}(S)}{|S|} - \sum_{S \in \Psi_{H_{-hj}} : j \in S} \frac{\Delta_{\mathfrak{P}_{H_{hj}}(w_T)}(S)}{|S|} \\ &= \rho_h^d(w_T, H_{-hj}) - \rho_j^d(w_T, H_{-hj}) \end{aligned} \tag{6.59}$$

Furthermore, by definition of Ψ_H, $S \in \Psi_H$ and $h \in S$ imply that $\overline{H}^{-1}(h) \subset S$.

Also, if $h \notin S$, then $S \in \Psi_H$ if and only if $S \in \Psi_{H_{-hj}}$.

Combining the stated properties it follows that for every player $i \in \overline{H}^{-1}(h)$ it holds that

$$\rho_i^d(w_T,H)-\rho_j^d(w_T,H)=\sum_{S\in\Psi_H:i\in S,\,h\notin S}\frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|}+$$
$$+\sum_{S\in\Psi_H:i,h\in S}\frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|}-\sum_{S\in\Psi_H:j\in S}\frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|}=$$
$$=\sum_{S\in\Psi_{H_{-hj}}:i\in S,\,h\notin S}\frac{\Delta_{\mathfrak{P}_{H_{-hj}}(w_T)}(S)}{|S|}+$$
$$+\sum_{S\in\Psi_H:h\in S}\frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|}-\sum_{S\in\Psi_H:j\in S}\frac{\Delta_{\mathfrak{P}_{H(w_T)}}(S)}{|S|}$$

This then leads to the conclusion that for every player $i\in\overline{H}^{-1}(h)$ it holds that

$$\rho_i^d(w_T,H)-\rho_j^d(w_T,H)=\sum_{S\in\Psi_{H_{-hj}}:i\in S,\,h\notin S}\frac{\Delta_{\mathfrak{P}_{H_{-hj}}(w_T)}(S)}{|S|}+$$
$$+\sum_{S\in\Psi_{H_{-hj}}:h\in S}\frac{\Delta_{\mathfrak{P}_{H_{-hj}}(w_T)}(S)}{|S|}-\sum_{S\in\Psi_{H_{-hj}}:j\in S}\frac{\Delta_{\mathfrak{P}_{H_{-hj}}(w_T)}(S)}{|S|}=$$
$$=\rho_i^d(w_T,H_{-hj})-\rho_j^d(w_T,H_{-hj})$$

With (6.59) we can now conclude that for every player $i\in\{h\}\cup\overline{H}^{-1}(h)$:

$$\rho_i^d(w_T,H)-\rho_j^d(w_T,H)=\rho_i^d(w_T,H_{-hj})-\rho_j^d(w_T,H_{-hj}).$$

This show the desired fairness property for w_T. For arbitrary games v with a strictly hierarchical permission structure H it can also be derived that

$$\mathfrak{P}(v,H)(S)=v(\delta_H(S))=\sum_{T\subset N}\Delta_v(T)\cdot u_T(\delta_H(S))=$$
$$=\sum_{T\subset N}\Delta_v(T)\cdot\mathfrak{P}_H(u_T)(S)\text{ and, similarly,}$$
$$\mathfrak{P}(v,H_{-hj})(S)=\sum_{T\subset N}\Delta_v(T)\cdot\mathfrak{P}_{H_{-hj}}(u_T)(S).$$

Finally, the additivity of the Shapley value now implies that indeed ρ^d satisfies disjunctive fairness.

Second, take an allocation rule $f:\mathcal{G}^N\times\mathfrak{H}_s^N\to\mathbb{R}^N$ that satisfies all of the six stated axioms. We will show that $f=\rho^d$.

Again let $w_T=C\cdot u_T$ for $T\subset N$ and some constant $C\in\mathbb{R}$. As in the proof of Theorem 6.24 we apply induction on the total number of authority relations in the hierarchical permission structure H:

Note that since H is strictly hierarchical, $\sum_{i\in N} \#H(i) \geqq n-1$.

If $\sum_{i\in N} \#H(i) = n-1$, then $\#H^{-1}(i) = 1$ for all $i \neq i_H$, and, thus, $\overline{H}(i) = H^-(i)$ for all $i \in N$. In that case $T \cap \left(\{i\} \cup \overline{H}(i)\right) \neq \varnothing$ for all $i \in H^-(T)$.[13] Also, $H^-(T) = \beta(T)$, where

$$\beta(T) = \left\{\, i \in H^-(T) \mid T \cap \left(\{i\} \cup \overline{H}(i)\right) \neq \varnothing \,\right\}$$

Using the weakly inessential player property, the necessary player property and weak structural monotonicity, we can then conclude that there is some constant $k \neq 0$ such that

$$f_i(w_T, H) = \begin{cases} k & \text{if } i \in H^-(T) \\ 0 & \text{otherwise.} \end{cases} \tag{6.60}$$

Efficiency then implies that $k = \frac{C}{\#H^-(T)}$, and thus $f(w_T, H) = \rho^d(w_T, H)$.

Proceeding by induction on the total number of authority relations $\sum_{i\in N} \#H(i)$ in H, we now assume that $f(w_T, H') = \rho^d(w_T, H')$ for all strictly hierarchical permission structures H' with $\sum_{i\in N} \#H'(i) < \sum_{i\in N} \#H(i)$.

Again using the construction in the proof of Theorem 6.24, we define recursively the level sets $\{L_k \mid k = 0, 1, 2, \cdots\}$ of H by $L_0 = \varnothing$ and for all $k \in \mathbb{N}$:

$$L_p = \left\{\, i \in N \setminus \cup_{t=0}^{p-1} L_t \;\middle|\; H(i) \subset \cup_{t=0}^{p-1} L_t \,\right\}$$

As before, denote $M > 0$ as the total number of level sets in H.

Let $k \geqq 0$ be such that $f_i(w_T, H) = k$ for all $i \in \beta(T)$.[14] Next we describe an algorithm to determine the allocated values $f_i(w_T, H)$ for all players $i \in N$ as a function of k:

Step 1: For all players $i \in L_1$ one of the following two conditions must hold:

(i) If $i \in N \setminus T$, then $f_i(w_T, H) = 0$.
(ii) If $i \in T$, then $f_i(w_T, H) = k$ since $i \in H^-(T)$.

Let $t = 2$.

Step 2: If $t = M + 1$, STOP. Otherwise, for every $i \in L_t$ one of the following three conditions has to be satisfied:

(i) If $i \in N \setminus H^-(T)$, then $f_i(w_T, H) = 0$.

[13] From the assumed special nature of the permission structure H it should be clear that $H^-(T)$ is the unique disjunctive authorizing cover of T in H.

[14] The existence of this constant k follows from the application of the weakly inessential player property, the necessary player property and weak structural monotonicity of the allocation rule f.

(ii) If $i \in \beta(T)$, then $f_i(w_T, H) = k$.

(iii) If $i \in H^-(T) \setminus \beta(T)$, then by definition of $H^-(T)$ and $\beta(T)$ there exists some $h \in \{i\} \cup \overline{H}(i)$ and some $j \in H(h)$ such that $\#H^{-1}(j) \geqq 2$. Disjunctive fairness then implies that

$$f_i(w_T, H) - f_i(w_T, H_{-hj}) = f_j(w_T, H) - f_j(w_T, H_{-hj}).$$

Using the induction hypothesis we then can write

$$f_i(w_T, H) = f_j(w_T, H) + \rho_i^d(w_T, H_{-hj}) - \rho_j^d(w_T, H_{-hj}) \qquad (6.61)$$

Since $j \in H^+(i)$ implies that $j \in L_q$ with $q < t$ it is already determined that $f_j(w_T, H)$ is a function of k. Thus, by (6.61) we have indeed determined $f_i(w_T, H)$ as a function of k.

Step 3: Let $t = t + 1$ and return to Step 2 of this procedure.

The given procedure determines for all players in a finite number of steps the values of $f(w_T, H)$ as a function of k. Efficiency then uniquely determines the value of k. Since the disjunctive permission value ρ^d satisfies these six axioms, it must therefore hold that $f(w_T, H) = \rho^d(w_T, H)$.

The proof of Theorem 6.25 is completed with the standard arguments already used at the conclusion of the proof of Theorem 6.24 to extend this conclusion from the class of proportional unanimity games to the class of all games $\mathcal{G}^N$. I omit the details of this derivation.

6.5 Discussion: More About the Disjunctive Restriction

From Theorem 6.18 for every cooperative game $v \in \mathcal{G}^N$ the disjunctive restriction on a weakly hierarchical or strict permission structure $H \in \mathfrak{H}_w^N$ is given by

$$\mathfrak{P}_H(v) = \sum_{S \in \Psi_H} \left\{ \sum_{T \in \mathfrak{A}_H^{-1}(S)} \Delta_v(T) + \sum_{T \in \widehat{\mathfrak{A}}_H(S)} \mu_S^H(T) \cdot \Delta_v(T) \right\} \cdot u_S.$$

In this appendix I show that for certain situations the multipliers $\mu_S^H(T)$ can be determined more precisely. This is stated as a proposition below. I first introduce some auxiliary concepts in the given context:

Definition 6.31 Let $S \subset N$ be some coalition. Take any $T \subset N$ such that $S \subset T$. The *essentiality index* of the coalition T with respect to coalition S in the strict permission structure $H \in \mathfrak{H}_w^N$ is the natural number

$$\eta_S^H(T) := \#\{R \in \mathfrak{A}_H(S) \mid R \subset T\}. \qquad (6.62)$$

The essentiality index $\eta_S^H(T)$ is defined simply as the number of disjunctive authorizing covers of coalition S that are contained in T.

It immediately follows that a coalition $S \subset N$ is disjunctively autonomous if and only if for every $T \subset N$ with $S \subset T$ it holds that $\eta_S^H(T) = 1$.

Proposition 6.32 *Let $H \in \mathfrak{H}_w^N$ be a strict permission structure on N and let $S \subset N$ be a coalition such that for every disjunctively autonomous cover $T \in \mathfrak{A}_H(S)$ of S there exists a player $i \in T$ such that $i \notin R$ for every alternative autonomous cover $R \in \mathfrak{A}_H(S) \setminus \{T\}$. Then for every coalition $R' \in \mathfrak{A}_H^*(S)$ which is the union of authorizing covers of S in H, it holds that*

$$\mu_{R'}^H(S) = (-1)^{\eta_S^H(R')+1} \in \{-1, 1\}. \tag{6.63}$$

Proof By the definition of the number $\mu_T(S)$ and the proof of Theorem 6.18, I only have to show that under the conditions as put on the coalition $S \subset N$ it holds that for every coalition $T \in \mathfrak{A}_H^*(S)$

$$\Delta_{w_S}(T) = (-1)^{\eta_S^H(T)+1}.$$

In order to prove this I use induction on the essentiality index $\eta_S^H(T)$. First suppose that $\eta_S^H(T) = 1$, then by Lemma 6.29 in the proof of Theorem 6.18

$$T \in \mathfrak{A}_H(S) \quad \text{and} \quad \Delta_{w_S}(T) = 1 = (-1)^2.$$

Let $K := \eta_S^H(T) \geqq 2$ and assume by the induction hypothesis that the formula as given above is true for all coalitions $R \in \mathfrak{A}_H^*(S)$ with $\eta_S^H(R) \leqq K - 1$. Obviously it holds that $w_S(T) = 1$ and so

$$\Delta_{w_S}(T) = 1 - \sum_{R \subset T : R \neq T} \Delta_{w_S}(R).$$

Since by Lemma 6.30 $\Delta_{w_S}(R) = 0$ for every coalition $R \notin \mathfrak{A}_H^*(S)$ we may restrict ourselves to the coalitions $R \subset T$ with $R \in \mathfrak{A}_H^*(S)$ and $R \neq T$.

By the induction hypothesis

$$\Delta_{w_S}(T) = 1 - \sum_{R \subset T : R \neq T \text{ and } R \in \mathfrak{A}_H^*(S)} (-1)^{\eta_S^H(R)+1}.$$

By the conditions as put on the coalition S it follows that T contains precisely $\frac{K!}{k!(K-k)!}$ subcoalitions $R \in \mathfrak{A}_H^*(S)$ with the property that $\eta_S^H(R) = k$, where $1 \leqq k \leqq n$. This in turn implies that

$$\Delta_{w_S}(T) = 1 - \sum_{k=1}^{K-1} \frac{K!}{k!\,(K-k)!} (-1)^{k+1} = \begin{cases} 1 & \text{for } K \text{ odd} \\ -1 & \text{for } K \text{ even} \end{cases}$$

since

$$\sum_{k=1}^{K-1} \frac{K!}{k!\,(K-k)!}\,(-1)^{k+1} = \begin{cases} 2 & \text{for } K \text{ even} \\ 0 & \text{for } K \text{ odd} \end{cases}$$

This completes the proof of the assertion.

6.6 Problems

Problem 6.1 Construct a proof of Lemma 6.4.

Problem 6.2 Let $H \in \mathcal{H}^N$ be some permission structure on the player set N. Assume the conjunctive approach to the exercise of authority in H. Prove the following assertions for some coalition $S \subset N$:

(a) $S \in \Gamma_H$ if and only if $H^{-1}(S) \subset S$ if and only if $H^-(S) \subset S$;
(b) $\gamma(S) = \{i \in S \mid H^{-1}(i) \subset S\}$;
(c) $\gamma(S) = S \setminus H^+(N \setminus S)$;
(d) $\gamma^-(S) = \{i \in N \mid H^+(i) \cap S \neq \varnothing\}$, and
(e) $\gamma^-(S) = S \cup H^-(S)$.

Problem 6.3 Let $N = \{1,2,3,4\}$. Construct a permission structure H on N such that there exist coalitions $S, T \subset N$ such that

(i) $\gamma_H(S) \cup \gamma_H(T) \subsetneqq \gamma_H(S \cup T)$ as well as
(ii) $\gamma_H^-(S \cap T) \subsetneqq \gamma_H^-(S) \cap \gamma_H^-(T)$.

Problem 6.4 Show the following property: A permission structure $H \in \mathcal{H}^N$ is strictly hierarchical if and only if for all players $i,j \in N$ there exists some player $h \in N$ such that $\{i,j\} \subset H^+(h)$.

Problem 6.5 Consider the notions of a hierarchical directed network as introduced in Chapter 5. Show through by constructing a sufficient number of examples that the class of strictly hierarchical permission structures $\mathfrak{H}_s^N$ is fundamentally different from the class of hierarchical directed networks $\mathfrak{H}^N$. In fact, show through these examples that $\mathfrak{H}_s^N \neq \mathfrak{H}^N$ with the properties that $\mathfrak{H}_s^N \cap \mathfrak{H}^N \neq \varnothing$, $\mathfrak{H}_s^N \setminus \mathfrak{H}^N \neq \varnothing$, $\mathfrak{H}^N \setminus \mathfrak{H}_s^N \neq \varnothing$ and $\mathfrak{H}_s^N \cup \mathfrak{H}^N \subsetneqq \mathfrak{H}_w^N$.

Problem 6.6 I recall some auxiliary properties of cooperative games that were discussed previously. Let $v \in \mathcal{G}^N$ be some cooperative game. Then:

(i) v is *monotone* if $v(S) \geqq v(T)$ for all $T \subset S$.
(ii) v is *superadditive* if $v(S) + v(T) \leqq v(S \cup T)$ for all $S, T \subset N$ with $S \cap T = \varnothing$.
(iii) v is *convex* if $v(S) + v(T) \leqq v(S \cup T) = v(S \cap T)$ for all $S, T \subset N$.
(iv) v is *balanced* if v has a nonempty Core.

Prove the following statements for any arbitrary permission structure $H \in \mathcal{H}^N$:

(a) If v is monotone, then its conjunctive restriction $\mathfrak{R}_H(v)$ is monotone as well.
(b) If v is superadditive, then its conjunctive restriction $\mathfrak{R}_H(v)$ is superadditive as well.
(c) If v is convex, then its conjunctive restriction $\mathfrak{R}_H(v)$ is convex as well.
(d) If v is balanced, then its conjunctive restriction $\mathfrak{R}_H(v)$ is balanced as well.[15]
(e) Suppose that $H \in \mathfrak{H}_s^N$ is strictly hierarchical. Show that if v is monotone, then its conjunctive restriction $\mathfrak{R}_H(v)$ is monotone, superadditive as well as balanced.

Problem 6.7 Consider the concepts from Problem 6.5 above. Let the cooperative game $v \in \mathcal{G}^N$ be monotone. Prove the following statements for any strict permission structure $H \in \mathfrak{H}_w^N$:

(a) The corresponding disjunctive restriction $\mathfrak{P}_H(v)$ of v is a monotone game as well.
(b) If v is additionally superadditive, then its disjunctive restriction $\mathfrak{P}_H(v)$ is superadditive as well.
(c) If v is monotone and convex, then its disjunctive restriction $\mathfrak{P}_H(v)$ is convex as well.
(d) If v is monotone and balanced, then its disjunctive restriction $\mathfrak{P}_H(v)$ is balanced as well.
(e) Suppose that $H \in \mathfrak{H}_s^N$ is strictly hierarchical. Show that if v is monotone, then its disjunctive restriction $\mathfrak{P}_H(v)$ is monotone, superadditive as well as balanced.

Problem 6.8 Consider the discussion of alternative organizations of the production situation described in Example 6.21. Instead of the constructed hierarchy H consider the hierarchical permission structure $H' \in \mathcal{H}^N$ given by

$$H'(i) = \begin{cases} P \cup \{b\} & \text{if } i = a \\ \{b\} & \text{if } i \in P \\ \emptyset & \text{if } i = b \end{cases}.$$

For the given hierarchy H' solve the following problems:

(a) Determine the exact values of ϱ and p for which the disjunctive hierarchy is strongly Pareto superior to the market mechanism with one-sided transaction costs.
(b) Consider the conjunctive hierarchies H and H'. Show that these hierarchies are conjunctively identical in the sense $\mathfrak{R}_H(v) = \mathfrak{R}_{H'}(v)$ for any game $v \in \mathcal{G}^N$. Furthermore, determine for which values of ϱ and p the conjunctive hierarchy

[15] HINT: Use the results from Chapter 2 on games with coalition structures, in particular a lattice of institutional coalitions.

H or H' is strongly Pareto superior to the market mechanism with one-sided transaction costs.

Problem 6.9 Consider the five examples following the first characterization of the conjunctive permission value constructed in Theorem 6.23. For each of these five examples, carefully check the properties satisfied by these allocation rules and determine that indeed these examples show the desired independence of the five axioms listed in Theorem 6.23.

Problem 6.10 This problem discusses a variation of Theorem 6.23 that is developed and shown in van den Brink and Gilles (1996) as well. First, I introduce two additional concepts on an allocation rule $f : \mathcal{G}^N \times \mathcal{H}^N \to \mathbb{R}^N$:

Strongly Inessential Player Property A player $i \in N$ is *strongly inessential* in the game with a permission structure (v, H) if i is a dummy player in the cooperative game v as well as $H(i) = \varnothing$.

Now for the allocation rule f it is assumed that for every strongly inessential player $i \in N$ in (v, H): $f_i(v, H) = 0$.

Inessential Relation Property Let $H \in \mathfrak{H}_w^N$ be weakly hierarchical or acyclic. For every strongly inessential player $i \in N$ in (v, H) it is assumed that

$$f_j(v, H) = f_j(v, H_{-i} \quad \text{for every } j \in N$$

where $H_{-i}(j) = H(j) \setminus \{i\}$ for all $j \in N$ is the permission structure in which all dominance relations with player i are eliminated.

Now with regard to these two introduced concepts answer the following questions.

(a) Is a strongly inessential player always necessarily a weakly inessential player? Provide a short proof if this is the case; otherwise provide a counter-example.
(b) Provide a proof of the following assertion:
An allocation rule $f : \mathcal{G}^N \times \mathfrak{H}_w^N \to \mathbb{R}^N$ is equal to the conjunctive permission value ρ^c on the class of games with an acyclic permission structure $\mathcal{G}^N \times \mathfrak{H}_w^N$ if and only if f satisfies efficiency, additivity, the strongly inessential player property, the inessential relation property, structural monotonicity, as well as the necessary player property.[16]
(c) The independence of the six axioms stated in (b) can be shown by application of the allocation rules developed in the main text and by considering two other allocation rules that address the strongly inessential player property and the inessential relation property:
    (i) Consider the allocation rule $f^6 : \mathcal{G}^N \times \mathfrak{H}_w^N \to \mathbb{R}^N$ given by

[16] Obviously, the required proof can be based on the proof of Theorem 6.23 provided in the appendix to this chapter.

$$f_i^6(v, H) = \frac{v(N)}{n}$$

where n is the number of players in N. Show that this allocation rule satisfies all properties of assertion (b) except the strongly inessential player property.

(ii) Consider $(v, H) \in \mathcal{G}^N \times \mathfrak{H}_w^N$. Define $U(H) = \{i \in N \mid H(i) \neq \varnothing\}$ and define $w_h \in \mathcal{G}^N$ by

$$w_H(S) = \begin{cases} v(S) & \text{if } S \subset U(H) \\ 0 & \text{otherwise.} \end{cases}$$

Now the allocation rule $f^7 : \mathcal{G}^N \times \mathfrak{H}_w^N \to \mathbb{R}^N$ is given by

$$f_i^7(v, H) = \varphi(w_H).$$

Show that the allocation rule f^7 satisfies all properties of assertion (b) except the inessential relation property.

Problem 6.11 Consider the axiomatization of the conjunctive permission value ρ^c given in Theorem 6.24. Next we determine that all six axioms listed are independent:

(a) Construct an allocation rule $f : \mathcal{G}^N \times \mathfrak{H}_s^N \to \mathbb{R}^N$ such that f satisfies all listed properties except efficiency.

(b) Let $f^1 : \mathcal{G}^N \times \mathfrak{H}_s^N \to \mathbb{R}^N$ be given by $f^1(v, H) = \varphi(v)$. Show that this allocation rule satisfies all listed properties except weak structural monotonicity.

(c) Let $f^2 : \mathcal{G}^N \times \mathfrak{H}_s^N \to \mathbb{R}^N$ be given by

$$f_i^2(v, H) = \begin{cases} v(N) & \text{if } i = i_H \\ 0 & \text{if } i \neq i_H. \end{cases}$$

Show that this allocation rule satisfies all listed properties except the necessary player property.

(d) Let $f^3 : \mathcal{G}^N \times \mathfrak{H}_s^N \to \mathbb{R}^N$ be given by $f^3(v, H) = \frac{v(N)}{n}$. Show that this allocation rule satisfies all listed properties except the weakly inessential player property.

(e) First, consider $T \subset N$ and $W_T \in \mathcal{G}^N$ be the game given by

$$W_T(S) = \begin{cases} 1 & \text{if } S \cap T \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $f^4 : \mathcal{G}^N \times \mathfrak{H}_s^N \to \mathbb{R}^N$ be given by

$$f^4(v,H) = \begin{cases} f^2(v,H) & \text{if } v = W_T \text{for some } T \subset N \text{with } |T| \geqq 2 \\ \rho^c(v,H) & \text{otherwise} \end{cases}$$

Show that this allocation rule satisfies all listed properties except additivity.

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